STUDIES IN STRONG DOUBLE DOMINATION IN GRAPHS

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in Mathematics

By

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CERTIFICATE

This thesis entitled **"STUDIES IN STRONG DOUBLE DOMINATION IN GRAPHS"**, submitted by ***P.Namasivayam*** for the award of Degree of **Doctor of Philosophy** in **Mathematics** to **Manonmaniam Sundaranar University** is a record of original research work done by him under our guidance and it has not been submitted for the award of any Degree, Diploma, Associateship, Fellowship of any University or Institution.

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DECLARATION

I do hereby declare that the thesis entitled **"STUDIES IN STRONG DOUBLE DOMINATION IN GRAPHS"** is the result of my original work carried out under the guidance of ***Dr.A.Subramanian***, Reader, Department of Mathematics, The M.D.T. Hindu College, Tirunelveli and that this work has not been submitted elsewhere for any other degree.

Tirunelveli,

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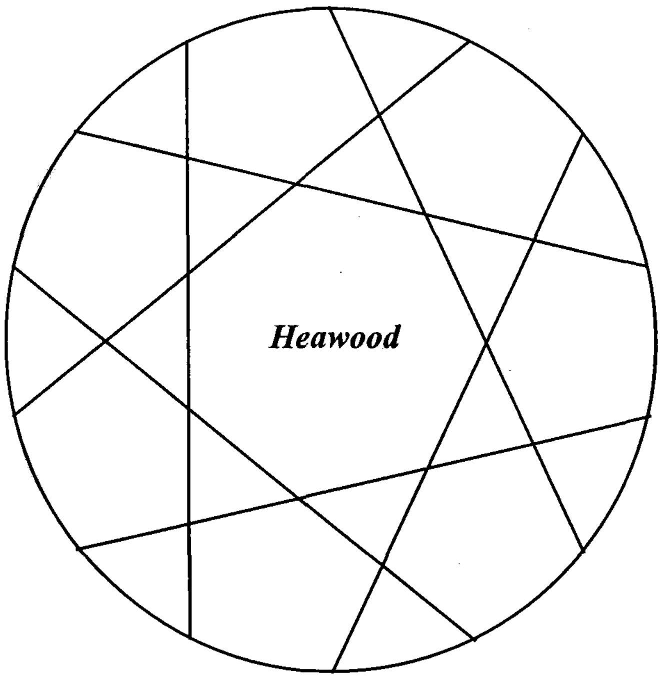
*Above all I thank **LORD ALMIGHTY** for his loving presence during my research.*

P. Namasivayam.



Reading makes a full man

Conference a ready man and

Writing an exact man.

- Francis Bacon.

INTRODUCTION

This thesis embodies the work done by the author under the guidance of Dr. A. Subramanian.

Domination in graphs has attracted the attention of many a mathematician due to its applicability in several areas like network problems, facility location problems, social network theory, school bus routing and set of representatives etc. A detailed analysis of domination and its applications is given in [34], Fundamentals of domination in graphs.

In ordinary domination for every vertex outside the dominating set there should be an adjacent vertex inside the dominating set. If we think of each vertex in a dominating set as a fileserver for a computer network then each computer in the network has direct access to a fileserver. It is sometimes reasonable to assume that this access be available even when one of the fileservers goes down. This has necessitated the existence of atleast two fileservers for access to each computer. Thus fault tolerance may compel the existence of more fileservers for access.

Double domination introduced by Harary and Haynes [19] serves as a model for the type of fault tolerance where each computer has access to atleast two fileservers and each of the fileservers has direct access to atleast one backup fileserver.

Sampathkumar and Pushpalatha [30] have introduced the concept of strong weak domination in graphs. This concept has application to traffic control, set of representatives with powers etc. A combination of the concepts of double domination and strong weak domination is the concept of domination strong domination where in for every vertex outside the dominating set, there are two vertices inside the dominating set, one of which dominates the outside vertex and the other strongly dominates the outside vertex. It has application in the formation of executive body in an administration. The executive body should be constituted in such a way that for each member of the organisation there should be atleast two members in the executive body who know them and in case of necessity the strength of one of them may be used to make the member to follow the rules. In communication network each computer has to access atleast two fileservers one of them being a " Powerful " fileserver.

Domination strong domination concept is the main theme of this thesis. A detailed study of this new type of domination has been made in the thesis. This thesis contains **five chapters** with a Bibliography at the end.

The first Chapter contains preliminary results needed for work in the study of this new type of domination.

The Second Chapter deals with the definition of domination strong domination (dom-strong domination or strong - double domination or dsd). The dom-strong domination number for standard graphs are found and the bounds for dom-strong domination number are also obtained. Nordhaus - Gaddum type results for dom-strong domination are attempted. k-dom-strong domination is defined and the results are obtained. The effects on γ_{dsd} (G) when G is modified by deleting a vertex is also discussed.

The Third Chapter deals with minimal dom-strong dominating set, excellent dsd set, split dsd set and dsd- domatic number.

Chapter four covers Independent dsd set, dsd irredundant and connected dsd sets.

The studies of complexity of double domination and dom-strong domination and fractional double domination constitute the **fifth chapter**. We proved that both the dd set and dsd sets are NP-Complete.

We conclude with some Open problems.

♦ ♦ ♦

CONTENTS

CHAPTER -I

1.1. Preliminaries :

In this chapter we collect the basic definitions and theorems which are needed for the subsequent chapters. For graph theoretic terminology, we refer to Harary [18] and Teresa Haynes [34].

1.1.1. Definition :

A graph G = (V, E) consists of a finite set denoted by V and a collection E of unordered pairs $\{u,v\}$ of distinct elements from V.

Each element of V is called a vertex and each element of E is called an edge.

The vertex set and the edge set of G are denoted by V(G) and E(G) respectively or simply V and E.

1.1.2. Definition :

If e = $\{u,v\}$ is an edge, we write e = uv, we say that e joins the vertices u and v ; u and v are adjacent vertices ; u and v are incident with e.

If two vertices are not joined then we say that they are non-adjacent.

If two distinct edges are incident with a common vertex then they are said to be adjacent to each other.

1.1.3. Definition :

The number of vertices, the cardinality of V is called the order of G and is denoted |V| or p. The cardinality of its edge set is called the size of G and is denoted by |E| or q.

A graph with p vertices and q edges is called a (p,q) – graph.

1.1.4. Definition :

A graph H is called a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A spanning subgraph of G is a subgraph H with V(H) = V(G). For any set S of vertices of G, the induced subgraph $\langle S \rangle$ is the maximal subgraph of G with vertex set S. Thus two vertices of S are adjacent in $\langle S \rangle$ if and only if they are adjacent in G.

1.1.5. Definition :

The degree of a vertex v in a graph G is the number of edges of G which are incident with v and is denoted by deg v. The minimum and maximum degrees of vertices of G are denoted by δ and Δ respectively.

1.1.6. Definition :

A vertex of G with degree 0 is called an isolated vertex; a vertex of G with degree 1 is called a pendant vertex. A vertex is called a support if it is adjacent to a pendant vertex.

1.1.7. Definition :

A graph G is said to be a regular – graph if every vertex of G is of same degree.

So a graph G is r- regular means every vertex of G is of degree r.

Any 3-regular graph is called a cubic-graph.

1.1.8. Definition:

Let G = (V, E) be a graph. The complement $\overline{G}$ of a graph G has $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

1.1.9. Definition :

The open neighborhood denoted by N (v) of a vertex v consists of the set of vertices adjacent to v, that is

$N(v) = \{ w \in V / \, v\, w \in E\}$.

Similarly the closed neighborhood of a vertex v denoted by N [v] is $N(v) \cup \{v\}$. (or) $N[v] = N(v) \cup \{v\}$.

1.1.10. Definition :

Let u and v be the vertices in a graph G. A walk u–v of length k is an alternating sequence of vertices and edges, namely $u = u_0, e_1, u_1, e_2 - - - - e_k, u_k = v$. If all the k edges are distinct then the walk is called a trail.

A walk in which all the vertices are distinct is called a path and if $u_0 = u_k$ but $u_1, u_2, ----u_{k-1}$ are all distinct then the trail is a cycle.

A walk is said to be open if u and v are distinct vertices; it is closed otherwise.

A path on n vertices is denoted by P_n and a cycle an n vertices is denoted by C_n.

1.1.11. Definition :

A graph G is a complete graph if every pair of vertices are adjacent. A complete graph on p vertices is denoted by K_p.

1.1.12. Definition :

A graph G is said to be connected if every pair of distinct vertices of G are joined by a path; otherwise G is disconnected.

A maximal connected subgraph of G is called a component of G.

So a connected graph has exactly one component, whereas a disconnected graph has atleast two components.

1.1.13. Definition :

The distance $d(u,v)$ between two vertices u and v in G is the length of a shortest $u-v$ path in G.

The diameter of G denoted by diam (G) of a connected graph is the maximum distance between two vertices of G.

1.1.14. Definition :

A graph which contains no cycles is called an acyclic graph or a forest. A connected acyclic graph is called a tree.

1.1.15. Definition :

A tree which yields a path when its pendant vertices are removed is called a caterpillar.

A spider is a tree which has atmost one vertex of degree ≥ 3.

1.1.16. Definition :

A subdivision of an edge $e = uv$ of a graph G is the replacement of the edge e by a path (u,ω,v). The graph obtained from G by subdividing each edge of G exactly once is called the subdivision graph of G.

1.1.17. Definition :

Let G = (V, E) be a graph. A subset D of V is said to be independent if no two vertices in D are adjacent.

The independence number β_o (G) is the maximum cardinality of an independent set in G. Note that every subset of an independent set is independent.

1.1.18. Definition:

The lower independence number *i* (G) is the minimum cardinality of a maximal independent set of G.

1.1.19. Definition :

The girth g (G) of a graph G is the length of a shortest cycle in G. The circumference c (G) of G is the length of a longest cycle.

1.1.20. Theorem:

For any graph G of size |E| = m, $\sum_{v \in V} \deg(v) = 2m$.

1.1.21. Definition :

A vertex and an edge are said to cover each other if they are incident. A set of vertices which cover all the edges of a graph G is called a vertex cover of G.

The smallest number of vertices in any cover for G is called its covering number and is denoted by $\alpha_o(G)$.

1.1.22. Definition :

Let G = (V, E) be a graph. A subset D of V is called a dominating set if every vertex $u \in V$ is either an element of D or is adjacent to an element of D.

A dominating set D is a minimal dominating set if no proper subset $D' \subset D$ is a dominating set.

The minimum cardinality of a dominating set is denoted by γ.

1.1.23. Definition :

Let G = (V, E) be a graph. A set $D \subseteq V$ is a strong dominating set of G if every vertex in V–D is strongly dominated by atleast one vertex in D.

1.1.24. Definition :

Let G = (V,E) be a graph. A set $D \subseteq V$ is a weak – dominating set of G if every vertex in V–D is weakly dominated by atleast one vertex in D.

Given two adjacent vertices u and v, u strongly dominates v if deg u $\geq$ deg v. Also v weakly dominates u if deg $v \leq$ degu.

The strong domination number γ_s (G) is the minimum cardinality of a strong dominating set of G.

Similarly the weak domination number γ_w (G) is the minimum cardinality of a weak dominating set of G.

Sampathkumar and Pushpalatha [30] introduced the notions of strong and weak domination in graphs.

1.1.25. Theorem :

A dominating set D is a minimal dominating set if and only if for every $u \in D$ one of the following holds:

(a) u is an isolate of D

(b) there exists a vertex $v \in V-D$ such that $N(v) \cap D = \{u\}$.

1.1.26. Definition :

A property P of sets of vertices is said to be hereditary if whenever a set S has property P, so does any proper subset $S' \subset S$.

A property P is Superhereditary if whenever a set S has property P, so does every proper superset $S' \supset S$.

1.1.27. Definition :

An independent set D is maximal independent if for every vertex $u \in V - D$, there is a vertex $v \in D$ such that u is adjacent to v.

An independent set D is maximal independent if and only if it is independent and dominating. This was first observed by Berge.

1.1.28. Definition :

Let D be a set of vertices and let $u \in D$. Then a vertex v is a private neighbor of u (with respect to D) if $N[v] \cap D = \{u\}$

1.1.29. Definition :

A set D is a total dominating set if $N(D) = V$ or equivalently, if for every vertex $v \in V$, there exists $u \in D$, $u \neq v$ such that u and v are adjacent.

The total domination number $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G.

1.1.30. Definition :

A set D of vertices is irredundant if for every vertex $v \in D$, $pn[v,D] \neq \phi$, that is every vertex $v \in D$ has atleast one private neighbor.

Here $pn[v,D]$ means private neighbor of v with respect to D.

ir (G) : The minimum cardinality of a maximal irredundant set in G.

IR (G): The maximum cardinality of an irredundant set in G.

1.1.31. Proposition:

A dominating set D is a minimal dominating set if and only if it is dominating and irredundant.

1.1.32. Proposition :

Every minimal dominating set in a graph G is a maximal irredundant set of G.

1.1.33. Theorem:

For any graph G,

$$\frac{\gamma(G)}{2} < ir(G) \leq \gamma(G) \leq 2ir(G) - 1$$

1.1.34. Definition :

A sequence $1 \leq a \leq b \leq c \leq d \leq e \leq f$ is said to be a domination sequence if there exists a graph G with ir (G) = a, γ (G) = b, i (G) = c, β_o(G)= d, Γ (G) = e and IR (G) = f.

1.1.35. Definition:

For a graph G with edges, the line graph L (G) is the graph whose vertices correspond to the edges of G and two vertices in L (G) are adjacent if and only if the corresponding edges in G are adjacent.

1.1.36. Definition :

The domatic number d (G) of a graph G is defined to be the maximum number of elements in a partition of V (G) into dominating sets.

Cockayne and Hedetniemi defined domatic number.

1.1.37. Definition :

The total domatic number d_t (G) is the largest number of sets in a partition of V into total dominating sets.

The connected domatic number d_c (G) of a graph G is the maximum number of sets in a partition of V into connected dominating sets.

Hedetniemi and Laskar defined connected domatic number.

1.1.38. Definition :

If there exists atleast one partition of V into independent dominating sets then G is called idomatic and the idomatic number id(G) equals the maximum number of sets in a partition of V into independent dominating sets.

1.1.39. Theorem:

For any graph G,

$$d(G) \geq \left\lfloor \frac{n}{n-\delta(G)} \right\rfloor \text{ and}$$

$$d_t(G) \geq \left\lfloor \frac{n}{n-\delta(G)+1} \right\rfloor$$

1.1.40. Theorem:

For any graph G, d (G) + d ($\overline{G}$) ≤ n+1 with equality if and only if G = K_n or $\overline{K_n}$.

1.1.41. Theorem :

For any graph G with δ (G) ≥ 1 and Δ (G) ≤ n – 2,

$d_t(G) + d_t(\overline{G}) \leq n - 1$.

1.1.42. Definition:

A Split graph is a graph G = (V,E) whose vertices can be partitioned into two sets V' and V" where the vertices in V' form a complete graph and the vertices in V" are independent.

1.1.43. Notation :

NP: Nondeterministic Polynomial time.

1.1.44. Theorem :

A Dominating set is NP – complete. This theorem was proved by David Johnson.

1.1.45. Definition :

Let f be the function defined by f :V (G) → [0,1] with $\sum_{u \in N(v)} f(u) \geq 1$ for every $v \in$ V (G). The fractional domination number $\gamma_f(G)$ is the minimum value of $\sum_{v \in V(G)} f(v)$, where the minimum is taken over all dominating functions f.

1.1.46. Theorem :

If G has n vertices and is k – regular then $\gamma_f(G) = \frac{n}{k+1}$

1.1.47. Result :

(a) For a cycle C_n, $\gamma_f(C_n) = \frac{n}{3}$

(b) For a complete graph K_n, $\gamma_f(K_n) = 1$.

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CHAPTER – II

In this chapter we introduce our concept Domination strong domination (Dom-strong domination or strong double domination or dsd) and Nordhaus- Gaddum type results are established.

2.1. DOM- STRONG DOMINATION IN GRAPHS

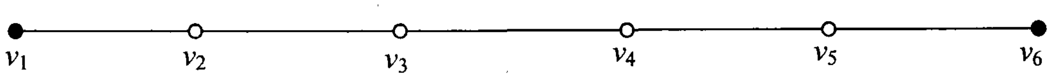
2.1.1. Definition :

Let G = (V, E) be a graph. A subset D of V is called a Domination strong domination set or dom-strong dominating set or dsd set or strong double dominating set if for every $v \in V - D$, there exists $u_1, u_2 \in D$ such that $u_1v, u_2v \in E(G)$ and $\deg u_1 \geq \deg v$.

The minimum cardinality of a Dom-Strong dominating set is called Dom-Strong domination number and is denoted by γ_{dsd}.

2.1.2. Example:

Consider P_6 :



Let V (P_6) = $\{v_1,v_2,v_3,v_4,v_5,v_6\}$ and the subset D = $\{ v_1,v_3,v_5,v_6\}$. Here D is a dsd set; For: $v_2 \in$ V–D, v_1 dominates v_2 and v_3 strong dominates v_2. Also for v_4 : v_3 dominates v_4 and v_5 strong dominates v_4.

The minimum cardinality of this case is $\gamma_{dsd} = 4$.

2.1.3. Observations :

(i) $\gamma_{dsd}(P_n) = \begin{cases} \left(\frac{n}{2}\right)+1, & \text{if n is even} \\ \left\lceil \frac{n}{2} \right\rceil, & \text{if n is odd.} \end{cases}$

(ii) $\gamma_{dsd}(K_n) = 2$

(iii) $\gamma_{dsd}(C_n) = \left\lfloor \frac{n+1}{2} \right\rfloor$

(iv) $\gamma_{dsd}(K_{m,n})$, (m > n) = n

(v) $\gamma_{dsd}(S) = |V(S)|$, S is a star (or) $\gamma_{dsd}(K_{1,n}) = n + 1$

(vi) $\gamma_{dsd}(W_{n+1}) = \left\lceil \frac{n}{3} \right\rceil + 1$

(vii) For a regular graph, $\gamma_{dsd} = \gamma_{dd\text{-}open}$

(viii) $\gamma_{dsd}(P) = 4$, where P is the Petersen Graph.

2.1.4. Theorem :

Let G be a graph with no isolates. Then $2 \leq \gamma_{dsd}(G) \leq n$ and the bounds are sharp.

Proof:

Since any dsd set has atleast two elements and atmost n elements, the theorem follows.

For a star, $\gamma_{dsd} = n$ and for K_n, $\gamma_{dsd} = 2$. Therefore the bounds are sharp. ■

2.1.5. Lemma :

If a vertex v has degree one then v must be in every dsd set of G.

That is every dsd set contains all pendant vertices.

Proof:

Let D be any dsd set of G. Let v be a pendant vertex with support say u. It $v \notin D$ then there must be two points say $x, y \in D$ such that x dominates u and y strong dominates u. Therefore x and y are adjacent to v. Then $\deg v \geq 2$, a contradiction, since v is a pendent vertex. So $v \in D$. Hence, the lemma. ■

2.1.6. Remark :

Support of a pendant vertex need not be in a dsd set.

2.1.7. Remark :

If the support u of a pendant vertex v is not in a dsd set then there exists a $x \in N(u)$ such that deg $x \geq$ deg u.

2.1.8. Theorem :

A connected graph G has V as its unique dsd set if and only if G is a star.

Proof :

If G is a star then V is the only dsd set. Conversely suppose V is the unique dsd set.

Suppose G is not a star.

Let x be a point with maximum degree, then all its neighbors are of degree one. (Otherwise $V - \{v\}$ is a dsd set). If $y \in V - N[x]$ then G is disconnected. So $\Delta = n - 1$, a contradiction. Hence, G is a star. ■

2.1.9. Theorem:

γ_{dsd} = 2 if and only if there exists y_1 and y_2 such that $\deg y_1 = \deg y_2 = \Delta$, $\deg y_1 \geq n-2$.

Proof:

Let there exists y_1 and y_2 satisfying the hypothesis. Let $D = \{y_1,y_2\}$. Let $x \in V - D$, then x is adjacent to both y_1 and y_2. Therefore $\deg x \leq \deg y_1$ and $\deg x \leq \deg y_2$. Therefore D is a dsd set.

Conversely, Let $D = \{y_1,y_2\}$ be a dsd set. Every point $x \in V-D$ is adjacent to both y_1 and y_2. Therefore $\deg y_1 \geq n-2$, $\deg y_2 \geq n-2$. Also $\deg x \leq \deg y_1$ or $\deg y_2$.

Suppose $\deg y_1$ and $\deg y_2 < \Delta$. Then there exists a $x \in V - D$ of degree Δ. Therefore D is not a dsd set. Hence, $\deg y_1 = \deg y_2 = \Delta$. If $\deg y_1 \neq \deg y_2$, then $\deg y_1 = n-1$ and $\deg y_2 = n-2$. Therefore y_1 and y_2 are adjacent. Therefore there exists a $x \in V - D$ such that x is not adjacent to y_2, a contradiction, since D is a dsd set. So $\deg y_1 = \deg y_2$. Hence, the theorem. ■

2.1.10. Theorem:

Let G be a graph without isolates and let there exist a γ_{dsd} set which is not independent. Then $\gamma(G) + 1 \leq \gamma_{dsd}(G)$.

Proof:

Let D be a γ_{dsd} set which is not independent. Let $x \in D$ be such that x is adjacent to some point of D. If $N(x) \cap (V-D) = \phi$ then as G has no isolates, $N(x) \cap D \neq \phi$. Hence $D - \{x\}$ is a dominating set.

Therefore $\gamma(G) \leq |D - \{x\}| = \gamma_{dsd}(G) - 1$.

Therefore $\gamma(G) + 1 \leq \gamma_{dsd}(G)$.

If $N(x) \cap (V-D) \neq \phi$ then for any $y \in N(x) \cap (V-D)$, there exists a $z \in D$ such that z is adjacent to y. As x is adjacent to some point of D, $D - \{x\}$ is a dominating set.

Therefore $\gamma(G) \leq |D - \{x\}| \leq \gamma_{dsd}(G) - 1$.

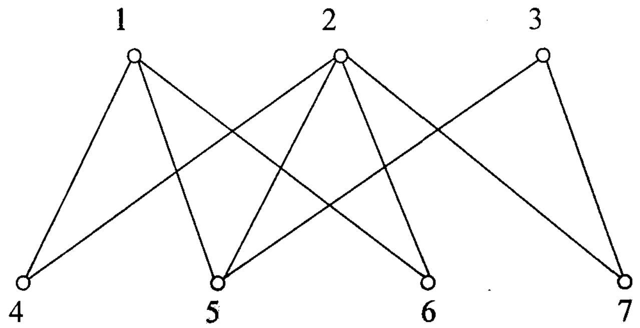
So $\gamma(G) + 1 \leq \gamma_{dsd}(G)$.

The bound is sharp, For: $K_{2,4}$ has $\gamma = 1$ and $\gamma_{dsd} = 2$. ■

2.1.11. Remark :

Let G have no isolates. If every γ_{dsd} set is independent then $\gamma = \gamma_{dsd}$ may not be true.

For example, consider the graph G:



$K_{3,4}$

Here every γ_{dsd} set is independent. But $2 = \gamma = \gamma_{dsd} - 1$.

Let G be a graph with no isolates. Let D be a γ_{dsd} set. Suppose D is independent and suppose there exists a subset S of D with $|S| \geq 2$ such that,

(i) $N(S) \subseteq N(D - S)$ and

(ii) a subset T of $V - D$ such that $|T| < |S|$ and $N(T) \supseteq S$ then $\gamma < \gamma_{dsd}$.

For: $T \cup (D - S)$ is a dominating set and

$\gamma = |T \cup (D - S)| < |D| = \gamma_{dsd}$. ■

2.2. k – Dom – Strong domination in graphs :

2.2.1. Definition :

Let G = (V, E) be a simple graph. A subset D of V is called a k – dom – strong dominating set of G, (k a positive integer) if for every $v \in V-D$, there exists two points $u_1, u_2 \in D$ such that $d(u_i, v) \leq k$, $i = 1,2$ and $d(u_2) \geq d(v)$.

The minimum cardinality of a k – dom – strong dominating set is denoted by γ_{kdsd} (G) and is called the k – dsd number.

2.2.2. Definition :

Let G = (V,E) be a graph. A set D of V is k-independent if for every $u, v \in D$, $d(u,v) > k$.

2.2.3. Observation :

$\gamma_k \leq \gamma_{kdsd} \leq \gamma_{dsd}$.

2.2.4. Observation:

When k = 1, we have the dom – strong domination.

2.2.5. Result:

Let G be a graph with no isolates and order of G = n $\geq$ 2 then $2 \leq \gamma_{kdsd}(G) \leq n$ and the bounds are sharp.

Proof:

Since any k – dsd set has atleast two elements and atmost n elements we have $2 \leq \gamma_{kdsd}(G) \leq n$ and the bounds are sharp, For: K_n has $\gamma_{kdsd} = 2$ and For $G = nk_2$, $\gamma_{kdsd} = 2n$ = order of G. ■

2.2.6. Remark:

$$\gamma_{kdsd}(K_{1,n}) = \begin{cases} 2, & \text{if } k \geq 2 \\ n+1, & \text{if } k = 1 \end{cases}$$

2.2.7. Observation:

A dsd set contains all the pendant vertices but a k– dsd set ($k \geq 2$) need not contain all pendant vertices.

2.2.8. Theorem:

A connected graph G of order ≥ 3 has V as its unique γ_{kdsd} – set if and only if k = 1 in which case G is a star.

Proof:

If k = 1, G has V as its unique γ_{kdsd} – set if and only if G is a star. Conversely, Let G have V as its unique γ_{kdsd} – set. Suppose k ≥ 2. Then G is not a star. Therefore diam (G) ≥ 3. Hence, there exists an induced path of length ≥ 3. Hence, all the vertices in this path are dominated by two points. Hence V is not a γ_{kdsd} – set, a contradiction. So k = 1. Hence, the theorem. ■

2.2.9. Observation:

If G is connected and if diam (G) ≤ k then $\gamma_{kdsd} = 2$.

2.2.10. Definition:

Let G = (V, E) be a graph. Let u ∈ V (G). Then k – degree of u is defined as k – deg (u) = $|\{v \in G: 1 \leq d(u,v) \leq k\}|$

2.2.11. Notation:

$\Delta_k = \max \{ k - \deg u : u \in V\}$

$\delta_k = \min \{ k - \deg u : u \in V\}$

$D_k(u) = \{ v \in G : 1 \leq d(u,v) \leq k\}$

$D_{wk}(u) = \{ v \in G : 1 \leq d(u,v) \leq k \text{ and } \deg u \geq \deg v\}$

The sk – degree (u) is defined as $|D_{wk}(u)|$

$\Delta_{sk} = \max \{ sk - \deg u : u \in V\}$

$\delta_{sk} = \min \{ sk - \deg u : u \in V\}$

2.2.12. Definition:

Two points u,v are said to be k – adjacent if $d(u,v) \leq k$.

2.2.13. Theorem:

$\gamma_{kdsd}(G) = 2$ if and only if there exists two non k– adjacent vertices y_1 and y_2 such that k–deg y_1 = k–deg y_2 = n–2 and sk–deg $y_1 = \Delta_{sk}$ or sk–deg $y_2 = \Delta_{sk}$.

Proof:

Let there exists y_1 and y_2 satisfying the hypothesis. Let $D = \{y_1, y_2\}$. Let $x \in V - D$, since k–deg y_1 = k–deg y_2 = n–2, x is k–adjacent to both y_1 and y_2. Also since sk–deg $y_1 = \Delta_{sk}$ or sk – deg $y_2 = \Delta_{sk}$ we get that deg x ≤ deg y_1 or deg y_2. So D is a kdsd–set. Since |D| = 2, D is a γ_{kdsd} –set. Conversely Let D = {y_1,y_2} be a γ_{kdsd} –set. Every point in V – D is k– adjacent to both $y_1$and y_2. Therefore k–deg $y_1 \geq n-2$ and k–deg $y_2 \geq n-2$. Also deg x ≤ deg y_1 or deg y_2. Suppose sk–degree y_1 and sk – degree y_2 are less than Δ_{sk}. Then there exists a x ∈ V–D of sk–degree Δ_{sk}. Therefore D is not a kdsd –set, a contradiction. So sk– deg $y_1 = \Delta_{sk}$ or sk–deg $y_2 = \Delta_{sk}$. ■

2.2.14. Definition:

Let G be a graph with no k–isolates. Let u∈G. u is called a k–isolate if for every $v \neq u \in G, d(u,v) > k$.

2.2.15. Theorem :

Let G be a graph without isolates. Let there exists a γ_{kdsd} –set which is not k–independent. Then $\gamma_k + 1 \leq \gamma_{kdsd}$.

Proof:

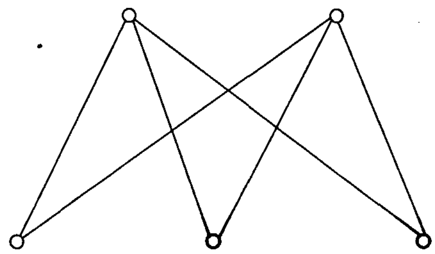
Let D be a γ_{kdsd} – set which is not k–independent. Then there exists x∈D such that x is k–adjacent to some point of D.

If $D_k(x) \cap \{V-D\} = \phi$ then as G has no k– isolates, $D_k(x) \cap D \neq \phi$. Hence $D - \{x\}$ is a k– dominating set. So $\gamma_k(G) \leq |D-\{x\}| = \gamma_{kdsd}(G)-1$ So $\gamma_k+1 \leq \gamma_{kdsd}$.

If $D_k(x) \cap \{V-D\} \neq \phi$ then for any $y \in (V-D) \cap D_k(x)$, there exists a $z \in D$ such that z is k– adjacent to y. Since x is k– adjacent to some point of D, $D - \{x\}$ is a k - dominating set and hence $\gamma_k(G) \leq |D-\{x\}| = \gamma_{kdsd}(G) - 1$. Hence $\gamma_k+1 \leq \gamma_{kdsd}$. ■

2.2.16. Remark:

The bound is sharp; For in G =



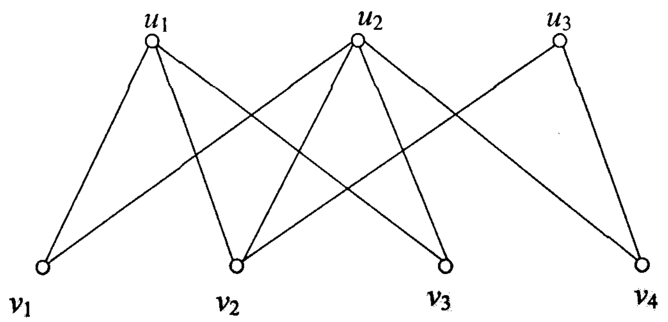
Here $\gamma_k(G) = 1$ and $\gamma_{kdsd}(G) = 2$.

2.2.17. Remark :

Let G have no k–isolates. Let D be a γ_{kdsd} – set. Suppose D is k–independent and there exists a subset S of D with $|S| \geq 2$ such that

(i) $D_k(S) \subseteq D_k(D-S)$

(ii) A subset T of V–D such that $|T| < |S|$, $D_k(T) \supseteq S$ then $\gamma_k < \gamma_{kdsd}$. For $T \cup (D-S)$ is a k– dominating set and $\gamma_k \leq |T \cup (D-S)| < |D| = \gamma_{kdsd}$. Consider the graph :



Here $\gamma_k = 2$, $\gamma_{kdsd} = 3$, $D = \{u_1,u_2,u_3\}$ $S = \{u_1,u_3\}$, $T = \{v_2\}$.

$\{u_1,u_2\}$ is a γ_k– set and $\{u_1,u_2,u_3\}$ is a γ_{kdsd} – set.

2.3. Complement graphs in Dom–Strong domination :

2.3.1. Definition:

Let G = (V, E) be a graph. The complement $\overline{G}$ of G is the graph with vertex set V (G) such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

2.3.2. Remark:

γ_{dsd} : dsd – number of G. So let us assume that $\overline{\gamma_{dsd}}$ denote the dsd – number of $\overline{G}$.

2.3.3. Remark:

Let x and y be the vertices of a graph G such that d (x,y) = diam G $\geq$ 3. There is no vertex in G adjacent to both x and y. Then every point in $\overline{G}$ is adjacent to both x and y.

2.3.4. Observation:

A graph G has dd = 3, (dd: double domination) if and only if there exists vertices u,v,w $\in$ G such that deg u = deg v = $\left\lceil \frac{2(n-3)}{3} \right\rceil$

$$\text{and deg } w = \begin{cases} (\deg u) - 1, \text{ if } 2\,(n-3) \equiv 2 \pmod 3 \\ (\deg u) - 2, \text{ if } 2\,(n-3) \equiv 1 \pmod 3 \end{cases}$$

2.3.5. Observation :

$$\gamma_{dsd}(K_n) + \overline{\gamma_{dsd}}(K_n) = n+1$$

Since $\gamma_{dsd}(K_n) = 2$ and $\overline{\gamma_{dsd}}(K_n) = n - 1$, we have the result. ■

2.3.6. Remark :

A point x $\notin$ V – D if

(i) x is adjacent to atmost one point.

(ii) deg x > deg y, for every y $\neq$ x.

That is if

(a) x is a pendant vertex or

(b) x is a strong vertex.

2.3.7. Theorem:

Let G be a connected graph. Then every vertex of G is either pendant or strong if and only if γ_{dsd} (G) = n.

Proof:

Let γ_{dsd} (G) = n. Suppose there exist a vertex v which is neither pendant nor strong then deg $v \geq 2$ and in N (v), there exists u such that deg $v \leq$ deg u. So V – {v} is a dsd set, which is a contradiction. (Since γ_{dsd} (G) = n). Hence every vertex of G is either pendant or strong. Conversely suppose every vertex of G is either pendant or strong. Then G is a star, because if x is a strong vertex then any neighbor of x is not strong, in such a case the neighbor is pendant. Hence γ_{dsd} (G) = n. ■

2.3.8. Result :

Let n > 3 and $\gamma_{dsd}(G) = n$ then $\gamma_{dsd}(G) + \overline{\gamma_{dsd}}(G) = n + 2$.

Proof:

Since $\gamma_{dsd}(G) = n$, G is a star. Hence $\overline{G}$ contains a complete graph and an isolated point, so $\overline{\gamma_{dsd}}(G) = 2$. Therefore

$$\gamma_{dsd}(G) + \overline{\gamma_{dsd}}(G) = n + 2.$$

2.3.9. Result:

$$\gamma_{dsd}(G) + \overline{\gamma_{dsd}}(G) \leq n + 2. \qquad \blacksquare$$

Next we examine the effects on $\gamma_{dsd}(G)$ when G is modified by deleting a vertex.

2.3.10. Definition:

Let G = (V,E) be a simple graph and let $V = V^{\circ} \cup V^{+} \cup V^{-}$

Define $V^{-} = \{v \in V: \gamma_{dsd}(G-v) < \gamma_{dsd}(G)\}$

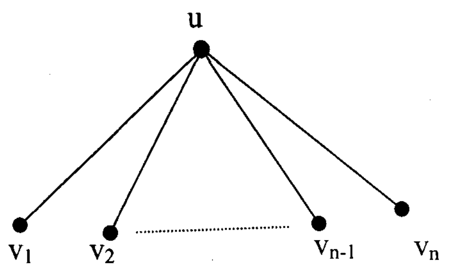
$V^{0} = \{v \in V: \gamma_{dsd}(G-v) = \gamma_{dsd}(G)\}$

$V^{+} = \{v \in V: \gamma_{dsd}(G-v) > \gamma_{dsd}(G)\}$

2.3.11. Definition:

(i) Let $G = K_{1,n}$

Let u be the centre and v_1, v_2 ----v_n be the pendant vertices.

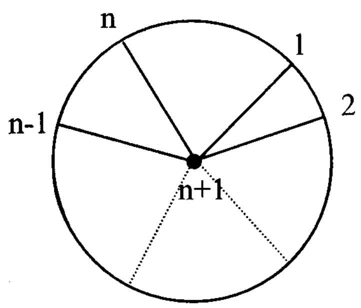


Then $\boxed{\gamma_{dsd}(K_{1,n} - v_i) < \gamma_{dsd}(K_{1,n})}$

(ii) Let $G = K_n$. Then

$\boxed{\gamma_{dsd}(K_n - u) = 2 = \gamma_{dsd}(K_n)}$ for any $u \in V(K_n)$, where $n \geq 3$.

(iii) Let $G = W_{n+1}$



$\gamma_{dsd}(W_{n+1} - u) > \gamma_{dsd}(W_{n+1})$,

if n is odd and $n \geq 9$ and $u = n+1$ is the centre of the wheel.

2.3.12. Definition:

Let $G = (V, E)$ be a simple graph.

Let $v \in V$. Let T be γ_{dsd} set of $G-v$, define

$S_T = \{u: u \in V-T, u\,v \in E(G), \deg_G v < \deg_G u, \deg_{G-v} x \leq \deg_{G-v} u$

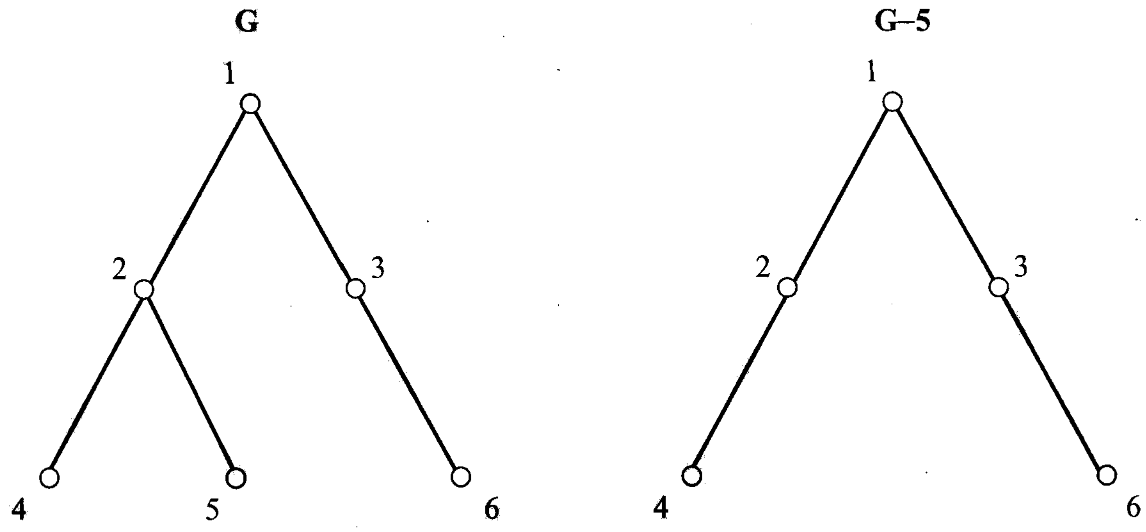
for all $x \in N(u) \cap T$, v is not adjacent to those vertices $y \in N(u) \cap T$

such that $\deg_{G-v} y = \deg_{G-v} u\}$.

(In this case $T \cup \{v\}$ is not a dsd set of G, since vertices in S_T will

not be dom strong dominated by $T \cup \{v\}$).

2.3.13. Illustration:



$\{1,2,4,6,5\}$ is a γ_{dsd} set of G.

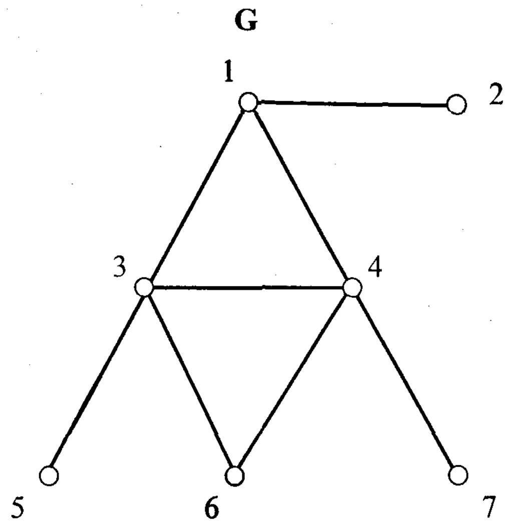
$T = \{1,4,6\}$ is a γ_{dsd} set of G–5.

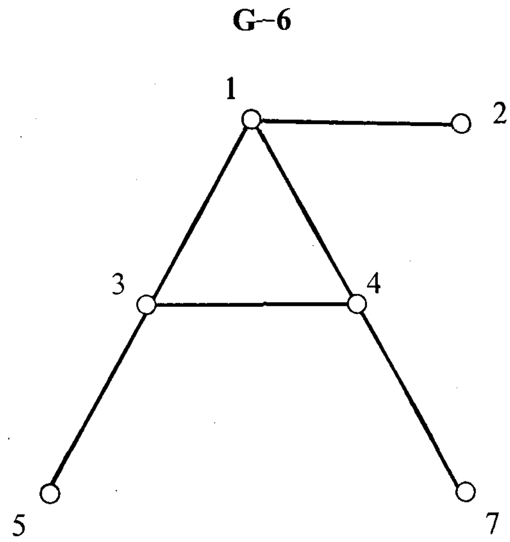
$S_T = \{2\}$

$T \cup \{v\}$ is not a dsd set of G.

But $T \cup \{v\} \cup S_T$ is a dsd set of G.

2.3.14. Illustration:





{2, 3,5,6,7} and {2,3,4,5,7} are dsd sets of G.

T = { 1,5,7,2} is a γ_{dsd} set of G–6.

T$\cup$ {v} = {1,5,7,2,6} is not a dsd set of G.

Here S_T = {3,4}

Therefore T$\cup$ {v}$\cup$ S_T = { 1,2,3,4,5,6,7} is a γ_{dsd} set of G.

2.3.15. Definition :

Let D be a subset of V and let $v \in$D. The Private neighbor of v with respect to D denoted by pn [v,D] is defined by

$$pn[v,D] = N[v] - N[D - \{v\}].$$

2.3.16. Definition :

The strong private neighbor of v denote by $pn_s[v,D]$ is defined by $pn_s[v, D]=N_w[v]-N_w[D-\{v\}]$ where

$N_w(x) = \{y \in v: y \in N(x) \text{ and } \deg y \leq \deg x\}$

2.3.17. Theorem :

Let $G = (V,E)$ be a simple graph. Let $v \in V^-$. Let for a γ_{dsd} set T of $G-v$, $S_T = \phi$. Then there exist a γ_{dsd} set D of G such that $pn[v,D] = \{v\}$ or $pn_s[v,D] = \{v\}$ or $|N(v) \cap D| = 1$, (In fact $D = T \cup \{v\}$ satisfies this property).

Proof:

Let $v \in V^-$. Then $\gamma_{dsd}(G-v) < \gamma_{dsd}(G)$. Let T be a γ_{dsd} set of $G-\{v\}$ such that $S_T = \phi$. Let $D = T \cup \{v\}$. Then D is a dsd set of G.

Therefore $\gamma_{dsd}(G) \leq |T| + 1$

Since $\gamma_{dsd}(G) > \gamma_{dsd}(G-v) = |T|$,

We get $\gamma_{dsd}(G) = |T| + 1$.

Since T is not a dsd set of G, either,

(i) $T \cap N(v) = \phi$ (or)

(ii) $|N(v) \cap T| = 1$ or $|N(v) \cap T| \geq 2$ and $N_s(v) \cap T = \phi$

Suppose (i) holds:

Suppose $T \cap N(v) = \phi$. Then $v \notin N[T]$

Therefore $v \in N[v] - N[T]$

Therefore $v \in pn[v,D]$ (Since $pn[v,D] = N[v] - N[D - \{v\}]$

$= N[v] - N[T]$)

Suppose $u \neq v$ and $u \in pn[v,D]$. Then $u \in N(v)$, $u \notin N[T]$.

Therefore $u \notin T$.

Since $u \neq v$, $u \in G - \{v\}$. Since $u \notin T$ and T is a dsd set of $G - v$, there exists $v_1, v_2 \in T$ such that v_1 dominates u and v_2 strong dominates u.

Therefore $u \in N(v_1)$ and $u \in N_w(v_2)$.

Therefore $u \in N(T)$, a Contradiction

Therefore $pn[v,D] = \{v\}$.

We have $pn_s[v, D] = N_w[v] - N_w[D - \{v\}]$

$= N_w[v] - N_w[T]$

Since $T \cap N(v) = \phi$, $v \notin N_w[T]$

Therefore $v \in N_w[v]$ and $v \notin N_w[T]$

Therefore $v \in pn_s[v,D]$.

Let $u \in N_w(v)$ and $u \notin N_w[T]$.

Now $u \notin T$ (If $u \in T$, then $u \in N_w(v)$, implies $u \in N(v)$,

so we get that $T \cap N(v) \neq \phi$, a contradiction to (i))

Since $u \neq v$, $u \in G - \{v\}$ and

Since $u \notin T$, there exists $v_1, v_2 \in T$ such that

v_1 dominates u and v_2 strong dominates u.

Therefore $u \in N_w(v_2)$

Therefore $u \in N_w(T)$, a contradiction,

Therefore $pn_s[v,D] = \{v\}$.

Suppose (ii) holds:

That is $|N(v) \cap T| = 1$ or $|N(v) \cap T| \geq 2$ and $N_s(v) \cap T = \phi$

Let $|N(v) \cap T| = 1$. Therefore $|N(v) \cap D| = 1$

Suppose $|N(v) \cap T| \geq 2$ and $N_s(v) \cap T = \phi$

Then $v \notin N_w[T]$.

(For: Suppose $v \in N_w[T]$. Then there exist a $u \in T$ such that $uv \in E$ and $\deg v \leq \deg u$, therefore $u \in N_s(v)$ and $u \in T$, therefore $u \in N_s(v) \cap T$ Hence $N_s(v) \cap T \neq \phi$, a contradiction)

Therefore $v \in N_w[v]$ and $v \notin N_w[T]$

Therefore $v \in N_w[v] - N_w[T]$

So $v \in N_w[v] - N_w[D - \{v\}]$

Therefore $v \in pn_s[v,D]$.

If $x \in pn_s[v,D]$ and $x \neq v$ then $x \in N_w(v)$ and $x \notin N_w[T]$

Therefore $x \notin T$. So there exists $v_1, v_2 \in T$ such that v_1 dominates x and v_2 strong dominates x

Therefore $x \in N_w(v_2)$.

Therefore $x \in N_w[T]$, a Contradiction.

So $pn_s[v,D] = \{v\}$. Hence the theorem. ■

2.3.18. Theorem :

Let $v \in V^+$. Then v is not an isolate of G, and v is not a pendant vertex of G.

Proof :

Suppose v is an isolate of G. Let D be γ_{dsd} set of G. Then $v \in D$ and $D-\{v\}$ is a γ_{dsd} set of $G-v$.

Therefore $\gamma_{dsd}(G-v) = |D|-1<|D| = \gamma_{dsd}(G)$.

$v \in V^-$, a contradiction.

So v is not an isolate of G.

Suppose v is a pendent vertex of G. Let D be a γ_{dsd} set of G. Then $v \in D$. Let u be the support of v. Then $(D-\{v\}) \cup \{u\}$ is a dsd-set of $G-v$.

Therefore $\gamma_{dsd}(G-v) \leq |D| = \gamma_{dsd}(G)$. So $v \notin V^+$, a contradiction

Hence v is not a pendant vertex of G.

2.3.19. Theorem :

Let $v \in V^+$. Then v belongs to every γ_{dsd} set of G.

Proof :

Suppose there exist a γ_{dsd} set D of G such that $v \notin D$. Then D is a dsd set of $G-\{v\}$. So $\gamma_{dsd}(G\text{-}v) \leq |D| = \gamma_{dsd}(G)$. So $v \notin V^+$, a contradiction.

So v belongs to every γ_{dsd} set of G. ■

2.3.20. Theorem:

Let $v \in V^+$. Then no subset of $V-N[v]$ with cardinality $\gamma_{dsd}(G)$ dom strong dominates $G-\{v\}$.

Proof:

Suppose there exist a subset S of $V-N[v]$ with cardinality $\gamma_{dsd}(G)$ dom strong dominates $G-\{v\}$.

Therefore: $\gamma_{dsd}(G-v) \leq |S| = \gamma_{dsd}(G)$. Then $v \notin V^+$, a contradiction. ■

♦ ♦ ♦

CHAPTER – III

In this chapter we determine the conditions of a minimal dsd – set, excellent dsd – sets and split dsd sets. dsd – domatic concept is also introduced.

3.1 Characterization of minimal dom-strong domination :

3.1.1. Observation:

Any superset of a dom–strong dominating set is a dom–strong dominating set. Hence dom–strong domination has super hereditary property.

Hence a subset D of V(G) is a minimal dom–strong dominating set if and only if it is a 1-minimal dom–strong dominating set.

3.1.2. Theorem:

Let G = (V, E) be a simple graph. Let $D \subseteq V$ be a dom–strong dominating set of G. D is a minimal dom–strong dominating set if and only if for every $u \in D$ one of the following holds:

(i) u is a pendant vertex of G.

(ii) u is an isolate in $\langle D \rangle$ or a strong isolate in $\langle D \rangle$ or $| N(u) \cap D | = 1$.

(iii) there exist a $v_1 \in V - D$ such that $N(v_1) \cap D = \{u\}$ or there exist a $v_2 \in V - D$ such that $N_s(v_2) \cap D = \{u\}$.

(iv) there exists a $v \in V - D$ such that $u \in N(v)$ and $|N(u) \cap D| = 2$.

Proof:

Let D be a minimal dsd set. Let $u \in D$. suppose u does not satisfy (i) to (iv). Then u is not a pendant vertex, so $|N(u)| \geq 2$. Also $N(u) \cap D \neq \phi$ and $N_s(u) \cap D \neq \phi$ and $|N(u) \cap D| \geq 2$.

Since u does not satisfy (iii) for every $v_1 \in V-D$, if $u \in N(v_1) \cap D$ then there exist $w \neq u$ such that $w \in N(v_1) \cap D$. Also for every $v_2 \in V-D$, if $u \in N_s(v_2) \cap D$, then there exist $w' \neq u$ such that $w' \in N_s(v_2) \cap D$. Consider $D - \{u\}$. Let $x \in (V-D) \cup \{u\}$. Suppose

$x = u$, Since $N(u) \cap D \neq \phi$, $N_s(u) \cap D \neq \phi$, and $|N(u) \cap D| \geq 2$, we get that $D - \{u\}$ dom–strong dominates u.

Suppose $x \neq u$. Then $x \in V-D$. Therefore there exists $y_1, y_2, \in D$, $y_1 \neq y_2$ such that $y_1 \in N(x) \cap D$ and $y_2 \in N_s(x) \cap D$. If $u \neq y_1$ and $u \neq y_2$ then $D- \{u\}$ Dom–strong dominates x. Since u does not satisfy (iv), we get that if $u \in N(v)$ for any $v \in V- D$ then $|N(v) \cap D| \neq 2$. Since D dom–strong dominates v, $| N(v) \cap D | \neq 1$. Therefore $|N(v) \cap D| \geq 3$.

Suppose $y_1 \neq u$. Then $u \in N(x)$. Therefore $|N(x) \cap D| \geq 3$. So there exist $w \in N(x) \cap D$, $w \neq u$ and $w \neq y_2$. Therefore $D-\{u\}$ dom–strong dominates x.

Suppose $y_2 = u$. Then $u \in N_s(x) \cap D. \subseteq N(x) \cap D$. So $|N(x) \cap D| \geq 3$. Therefore there exist $w \in N(x) \cap D$, $w \neq u$ and $w \neq y_1$. Therefore $D-\{u\}$ dom–strong dominates x. So $D-\{u\}$ is a dsd–set, a contradiction, since D minimal by assumption. Conversely, to show that D is minimal dsd–set, it is enough if we prove that D is a 1– minimal dsd–set.

Let D be dsd–set. Suppose $u \in D$ satisfies one of the following conditions. Consider $D-\{u\}$. If u satisfies (i) then $D-\{u\}$ is not a dsd–set. If u satisfies (ii) then D–{u} is not a dsd–set. Similarly if u satisfies (iii) or (iv) then also D–{u} is not a dsd–set. Hence, D is the minimal dsd set.

Hence the theorem. ■

3.1.3. Note:

There does not exists $v_1, v_2 \in D$ with $v_1 \neq v_2$ such that $v_1 u, v_2 u \in E$, $d(v_2) \geq d(v_1)$.

3.1.4. Note:

For any $w \in V-D$, there exist $v_1 \neq u$, $v_1 \in D$ such that $v_1 \in N(w)$ and $v_2 \in D$, $v_2 \neq u$, $v_2 \neq v_1$ such that $v_2 \in N(w)$ and $d(u) \geq d(w)$ is not true.

3.1.5. Remark:

Let H be a spanning subgraph of G. Then $\gamma_{dsd}(H) \geq \gamma_{dsd}(G)$.

For: any dsd set of H is also a dsd set of G.

3.1.6. Remark:

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. $G_1 + G_2$ is defined as a graph with vertex set $V_1 \cup V_2$ and edge set

$$E_1 \cup E_2 \cup \{uv / u \in V_1, v \in V_2\}$$

Let $|v_1| = p_1 \geq 2$ and $|v_2| = p_2 \geq 2$.

Case (i). Let $\Delta(G_1) = \Delta(G_2) \leq \min\{p_1-2, p_2-2\}$

Subcase (i): Suppose $p_1 < p_2$. A point u of degree $\Delta(G_1)$ in G_1 has degree $\Delta(G_2) + p_2$ in G_1+G_2. Hence u strong dominates all points of G_2.

Consider a γ_s- set of G_1. Clearly $\gamma_s \geq 2$. Then this set is a $\gamma_{dsd}-$ set for G_1+G_2. So $\gamma_{dsd}(G_1+G_2) = \gamma_s(G_1)$ or $\gamma_s(G_1) + 1$. According as a γ_s- set of G_1 is a dsd set of G_1 or no γ_s- set of G_1 is a dsd set of G_1.

Similar proof for $p_2 < p_1$.

Subcase (ii) : $p_1 = p_2$. Then take a point u in G_1 of degree $\Delta(G_1)$ in G_1 and a point v in G_2 of degree $\Delta(G_2)$ in G_2. Then {u,v,u',v'} with $u' \in V(G_1)$ and $v' \in V(G_2)$ is a dsd-set for G_1+G_2. Therefore $\gamma_{dsd}(G_1+G_2) = 4$.

Case (ii): let $\Delta(G_1) = p_1-1 < \Delta(G_2)$. Then a point u with degree p_1-1 in G_1 has degree $p_1 + p_2 - 1$ in G_1+G_2. If there are two vertices of degree p_1-1 in G_1 or if there exist a point in G_2 of degree p_2-1 in G_2 or if there exists two points of degree p_2-1 in G_2 then $\gamma_{dsd}(G_1+G_2) = 2$.

Suppose there exist a unique point of degree p_1-1 in G_1 and no point of degree p_2-1 in G_2 then {u,v',u'} is a dsd set of G_1+G_2 and hence $\gamma_{dsd}(G_1+G_2) = 3$.

Case (iii) $\Delta(G_2) = p_2-1$, Similar proof as in case (ii)

Case (iv): $\Delta(G_1) \leq p_1 - 2$, $\Delta(G_2) \leq p_2 - 2$

Consider Max $\{\Delta(G_1) + p_2, \Delta(G_2) + p_1\}$.

Suppose Max $\{\Delta(G_1) + p_2, \Delta(G_2) + p_1\} = \Delta(G_1) + p_2$. Then a point u of G_1, of degree $\Delta(G_1)$ in G_1, strong dominates all points of G_2.

Let $A = \{u_1 = u, u_2, \text{----} u_k\}$ be a γ_s–set of G_1. Then $\gamma_{dsd}(G_1+G_2) = \gamma_s(G_2)$ or $\gamma_s(G_2) + 1$. According as a γ_s– set of G_1 is a dsd set of G_1 or no γ_s - set of G_1 is a dsd set of G_1.

3.1.7. Remark:

For any positive integer $n \geq 2$, there exist a graph G such that $\gamma_{dsd}(G) = n$. (Take $G = K_{m,n}$ $(m>n)$. Then $\gamma_{dsd}(G) = n$)

3.1.8. Remark:

For a cube Q_n : when n=1, $Q_1 = K_2$ and hence $\gamma_{dsd}(Q_1)=2$.

When n=2, $Q_2 = C_4$ and hence $\gamma_{dsd}(Q_2) = 2$.

3.1.9. Theorem:

Let G be (p,q) graph. Let u be a vertex G of degree $\Delta(G)$. If u satisfies the property that for any $v \in N(u)$, $N(v) \cap (V - N[u]) \neq \phi$ then $\gamma_{dsd}(G) \leq p - \Delta$.

Proof:

Let $D = V - N(u)$. Then for any $v \in V - D$, v is strongly dominated by u. Also by hypothesis there exist $w \neq u$ in $V - D$ such that v is adjacent to w. Hence D is a dsd set.

So $\gamma_{dsd}(G) \leq |D| = p - \Delta(G)$. ■

3.1.10. Observation :

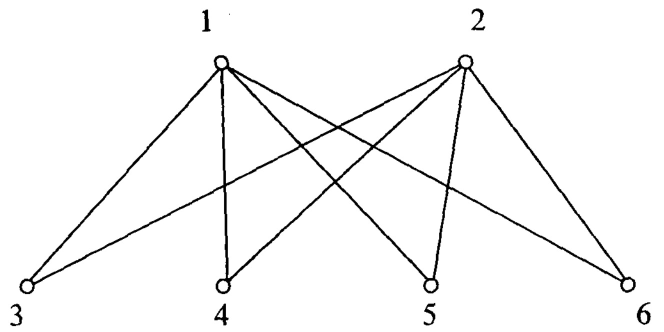
For a star $K_{1,n}$, $\Delta = n$. $p - \Delta = (n+1) - n = 1$ and $\gamma_{dsd}(K_{1,n}) = n + 1$.

So $\gamma_{dsd}(K_{1,n})$ is different from $p - \Delta$. So $\gamma_{dsd}(K_{1,n}) > p - \Delta$.

3.1.11. Observation :

We find examples for which $\gamma_{dsd} < p - \Delta$ and $\gamma_{dsd} = p - \Delta$.

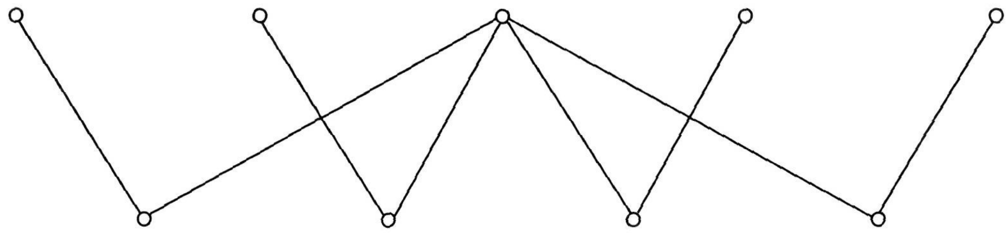
Consider the graph:



$K_{2,4}$

Here $\gamma_{dsd} = 2$. Also $p = 6$, $\Delta = 4$, so $p - \Delta = 2$. Hence $\gamma_{dsd} = p - \Delta$.

Consider the graph:



(H)

$\Delta = 4$, $p = 9+0$ (H) with Δ (H) ≤ 4, $\gamma_{dsd} = 5$. $p-\Delta = 5+0$ (H).

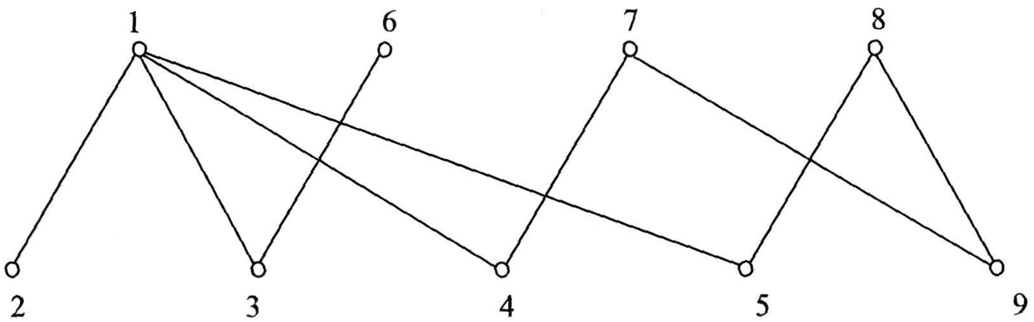
So $\gamma_{dsd} \leq 5 + 0$ (H). Hence $\gamma_{dsd} < p - \Delta$.

3.1.12. Observation:

The condition in the above theorem is not sufficient. That is if γ_{dsd} (G) $\leq p - \Delta$ then the following need not be true.

"There exist a vertex u of degree Δ such that for any $v \in N(u)$, $N(v) \cap (V - N[u]) \neq \phi$".

For: Consider G:



$p = 9$, $\Delta = 4$, so $p-\Delta = 5$.

{1,2,6,7,8} is a dsd-set. Thus $\gamma_{dsd} = p - \Delta = 5$.

But $2 \in N(1)$ and 2 is not adjacent to any point of $V - N[1]$.

3.1.13. Proposition:

$p-\frac{q}{2}+\frac{q_0}{2} \leq \gamma_{dsd}(G)$, where G is a (p,q) graph and q_0 = min q $(\langle D \rangle)$

D ∈ {minimal dsd sets of G}.

Proof:

Let D be a minimal dsd set of G. Then for any point u ∈ V–D, there exist atleast two edges from u to D and hence there exist atleast 2 | V–D| edges from V–D to D. Then the number of edges of G namely,

$q \geq 2\,|V-D| + q_0$

$= 2\,|V| - 2\,|D| + q_0$

or $2\,|D| \geq 2\,|V| - q + q_0$

so $2\,|D| \geq 2p - q + q_0$

or $|D| \geq p-\frac{q}{2}+\frac{q_0}{2}$

or $\gamma_{dsd}(G) \geq p-\frac{q}{2}+\frac{q_0}{2}$

or $p-\frac{q}{2}+\frac{q_0}{2} \leq \gamma_{dsd}(G)$, ■

3.1.14. Remark:

For $K_{1,n}$: $\gamma_{dsd} = n+1$, $p = n+1$, $q = n$, $q_0 = n$.

So $p - \frac{q}{2} + \frac{q_0}{2} = n+1 - \frac{n}{2} + \frac{n}{2} = n+1 = \gamma_{dsd}$

Thus the lower bound is attained.

3.1.15. Theorem:

For any (p,q) graph G with γ_{dsd} (G) = a, a=2 (p–q) if and only if $G = \frac{a}{2} K_2$.

Proof:

Let a = 2 (p–q).

Suppose G has t components. Then $2t \leq \gamma_{dsd}(G) = a$

Suppose 2t < a.

$q(G) \geq p - t > p - \frac{a}{2}$.

(or) $2q > 2p - a$

(or) $2p - a < 2q$

(or) $2p - 2q < a$ implies that $2(p-q) < a$.

(or) $a > 2(p-q)$ a contradiction, since $a = 2(p-q)$.

Then G has exactly $\frac{a}{2}$ components.

Let $G_1, G_2, \ldots.. G_{\frac{a}{2}}$ be the components of G.

Claim: each G_i is a star.

Suppose G_1 is not a star. Since G_1 is connected and not a star it follows that diam $G_1 \geq 3$ or G_1 is a graph having a cycle such that $\gamma_{dsd}(G_1) = 2$.

Let G_1 be a graph with diam ≥ 3. Then $\gamma_{dsd}(G_1) \geq 3$. So $\gamma_{dsd}(G) = \sum_{i=1}^{a/2} \gamma(G_i) \geq 2\left(\frac{\gamma_{dsd}}{2}\right) + 1 > \gamma_{dsd}(G_i), a$ contradiction.

If G_1 is a graph having a cycle such that $\gamma_{dsd}(G_1) = 2$ then $q(G_1) > p(G_1) - 1$. (Since G_1 is having a cycle.)

$$q(G) = q(G_1) + \sum_{i=2}^{a/2} q(G_i)$$

$$> p(G_1) - 1 + \sum_{i=2}^{a/2} (p(G_i) - 1) = p - \frac{a}{2}$$

so $q > p - \frac{a}{2}$ implies that

$\frac{a}{2} > p - q$ a contradiction.

So each G_i is a star.

Since $\gamma_{dsd}(G_i) = 2$, G_i is K_2. Thus $G = \frac{a}{2}K_2$.

3.1.16. Proposition:

$$\gamma_{dsd}(G) \geq \left\lceil \frac{2p}{\Delta+2} \right\rceil$$

Proof:

Every vertex in V–D contributes 2 to the degree sum of vertices of D. So $2|V-D| \leq \sum_{u\in D} d(u)$ where D is a dsd set.

So $2|V-D| \leq \sum d(u) \leq \gamma_{dsd}\Delta$

(or) $2|V-D| \leq \gamma_{dsd}\Delta$

(or) $2(|V|-|D|) \leq \gamma_{dsd}\Delta$

(or) $2p - 2\gamma_{dsd} \leq \gamma_{dsd}\Delta$

(or) $\gamma_{dsd}(\Delta+2) \geq 2p$

(or) $\gamma_{dsd} \geq \frac{2p}{\Delta+2}$

(or) $\gamma_{dsd}(G) \geq \left\lceil \frac{2p}{\Delta+2} \right\rceil$

3.1.17. Remark:

For K_n, $\gamma_{dsd}(K_n) = 2$.

$$\left\lceil \frac{2p}{\Delta+2} \right\rceil = \left\lceil \frac{2n}{n+1} \right\rceil = 2,$$

$$\text{that is } \gamma_{dsd} = \left\lceil \frac{2p}{\Delta+2} \right\rceil$$

So the bound is sharp.

3.1.18. Proposition:

For any given integer n, there exist a graph G with

$$\gamma_{dsd}(G) - \left\lceil \frac{2p}{\Delta+2} \right\rceil = n.$$

Proof:

Let $G = K_{1,n+1}$, $n \geq 1$.

Here $\gamma_{dsd}(G) = n+2$

$$\left\lceil \frac{2p}{\Delta+2} \right\rceil = \left\lceil \frac{2(n+2)}{n+3} \right\rceil = \left\lceil \frac{2n+4}{n+3} \right\rceil = \left\lceil \frac{2n+6-2}{n+3} \right\rceil$$

$$= \left\lceil \frac{2(n+3)-2}{n+3} \right\rceil = \left\lceil 2 - \frac{2}{n+3} \right\rceil = 2$$

$$\therefore\ \gamma_{dsd}(G) - \left\lceil \frac{2p}{\Delta+2} \right\rceil = n+2-2 = n.$$ ■

3.1.19. Definition:

A subset D of V is Dom– strong independent if

(i) D is independent.

(ii) for every point u of D there exists $v_1, v_2 \in V-D$ such that v_1 is adjacent to u and v_2 strong dominates u.

β_{dsd} denote the maximum cardinality of such a set D.

If no such a set exists then $\beta_{dsd} = 0$.

3.1.20. Proposition:

$\gamma_{dsd} \leq p - \beta_{dsd} + p_o$, where p_o is the number of isolates.

Proof:

If $\beta_{dsd} = o$ then $\gamma_{dsd} \leq p + p_o$. Let G' = G – {isolates of G}. For G' there exists a β_{dsd} – set D and hence V –D is a dsd set.

So $\gamma_{dsd}\ (G') \leq |V\ (G') - D| = (p - p_o) - \beta_{dsd}\ (G')$

Now $\gamma_{dsd}\ (G) = \gamma_{dsd}\ (G') + p_o$

$\leq (p - p_o) - \beta_{dsd}\ (G') + p_o$

$= p - p_o - (\beta_{dsd}\ (G) - p_o) + p_o$

$= p - \beta_{dsd}\ (G) + p_o$

So γ_{dsd} (G) $\leq$ p – β_{dsd} (G) + p_o

Hence the result. ■

3.1.21. Theorem:

Let G be a graph. Then γ_{dsd} (G) = n if and only if every component of G is a star.

Proof:

If every component of G is a star then γ_{dsd} (G) = |V(G)| = n. Conversely suppose γ_{dsd} (G) = n. Suppose there exist a component of G which is not a star. Let H be the component. Then H is a connected graph which is not a star. Let u be a point of degree Δ in H. Let $v_1, v_2 \ldots . v_\Delta$ be the points of H adjacent to u. If each v_i is pendant then H is a star, a contradiction. Therefore there exist a vertex v_i adjacent to u which is not pendant. Let v_i be adjacent to w ≠ u. Then V (H) – {v_i} is a dsd set. Therefore γ_{dsd} (H) ≤ |V(H)| – 1. So γ_{dsd} (G) ≤ n – 1, a contradiction, since γ_{dsd} (G) = n. Hence every component of G is a star.

■

3.1.22. Theorem:

Let G be a graph. Then $\gamma_{dsd}(G) \leq n - \Delta + k$, where k is the number of pendant vertices in G.

Proof:

Let u be a point with degree Δ. Let $v_1, v_2, \ldots v_\Delta$ be the neighbors of u. Let $v_1, v_2, \ldots v_s$ $(s \leq \Delta)$ be the points in N (u) which are pendant vertices. Then by theorem 3.1.21, $\gamma_{dsd}(G) \leq n - (\Delta - k) = n - \Delta + k$. ■

3.1.23. Remark:

Let G be a graph with maximum degree Δ. Let u be a point of G of degree Δ with s pendant neighbors. Then $\gamma_{dsd}(G) \leq n - \Delta + s$

Proof:

By proceeding as in theorem 3.1.22, we get $\gamma_{dsd} \leq n - r$, where $r = \Delta - s$. So $\gamma_{dsd}(G) \leq n - \Delta + s$. ■

3.1.24. Theorem:

Let G be a graph. Then $\gamma_{dsd} \leq n - \frac{1}{2}(\Delta - s + t)$.

Proof:

Let u be a point of degree Δ. Let $v_1, v_2, \ldots v_\Delta$ be the neighbors of u. Let $v_1, v_2 \ldots v_s$, $(s \leq \Delta)$ be the points in N (u) which are pendant vertices. Let $\{u_1, u_2 \ldots u_t\}$ be the 't' points which have adjacent points in $V - N[v]$ (= S). Then $S \cup \{v, v_1, v_2 \ldots v_s\} \cup D$ is a dsd set of G where D is a minimum dominating set of

$$\langle N(v) - \{v_1, v_2, \ldots. v_s, u_1, u_2, \ldots\ldots u_t\}\rangle$$

So $$\gamma_{dsd}(G) \leq |S| + s + 1 + \frac{1}{2}|N(v) - \{v_1, v_2 \ldots v_s, u_1, u_2, \ldots u_t\}|$$

$$= n - \Delta - 1 + s + 1 + \frac{1}{2}(\Delta - s - t)$$

$$= n - \Delta + s + \frac{1}{2}(\Delta - s - t)$$

$$So\ \gamma_{dsd}(G) \leq n - \frac{1}{2}(\Delta - s + t)$$ ■

3.1.25. Remark:

For any graph G,

$\Delta \leq 2n - 2\,\gamma_{dsd} + s - t$

That is $\Delta \leq 2n - 2\,\gamma_{dsd} + k$, where k is the number of pendant vertices of G.

3.1.26. Corollary:

When G is a star $K_{1,n}$, the equality is reached.

Here k = n and 'n' = n+1

So $\Delta \leq 2n - 2\,\gamma_{dsd} + k$ implies that

$\Delta \leq 2\,(n+1) - 2\,(n+1) + n = n.$

$\therefore\ \Delta = n.$

3.1.27. Corollary:

When $G = K_n$, $n \geq 3$.

$k = 0$, 'n' = n. $\Delta = n - 1$.

So $\Delta \leq 2n - 2\,\gamma_{dsd} + k$ gives $2\,\gamma_{dsd} \leq 2n - \Delta + k$

So $\gamma_{dsd} \leq n - \frac{\Delta + k}{2}$

Hence $\gamma_{dsd} \leq n - \frac{(n-1)+0}{2}$

So $\gamma_{dsd} \leq \frac{n+1}{2}$

3.1.28. Theorem:

Let G be a graph with δ (G) $\geq$ 2. Then G has atleast two minimal dsd sets.

Proof:

Since δ (G) $\geq$ 2, |V(G)| $\geq$ 3. Also G has atleast two non-strictly strong vertices say x,y. Then V−{x}, V−{y} are dsd sets of G. Let D,D' be minimal dsd sets of G contained in V−{x} and V−{y} respectively. Then D $\neq$ D'. Hence the theorem. ■

3.1.29. Corollary:

If G has a unique minimal dsd set then G has a pendant vertex or G = $\overline{K_n}$. ■

3.1.30. Theorem:

A graph G has a unique minimal dsd set if and only if every non–strong vertex of G is either a pendant vertex of G or is adjacent to a strong vertex of G.

Proof:

Suppose G has a unique minimal dsd set. Suppose u is a non strong vertex which is neither a pendant nor adjacent to any strong vertex of G. Then clearly $d(u) \geq 2$. Therefore $V-\{u\}$ is a dsd set containing a minimal dsd set say D. Since $u \notin D$ and since D is a dsd set there exists $v_1, v_2 \in D$ such that $uv_1, uv_2 \in E(G)$ and $d(v_2) \geq d(u)$. Since $d(u) \geq 2$, $d(v_2) \geq 2$. By hypothesis v_2 is not a strong vertex. Therefore $V-\{v_2\}$ is a dsd set containing a minimal dsd set say D'. Since $v_2 \in D$ and $v_2 \notin D'$, $D \neq D'$. Therefore there exists two minimal dsd sets, a contradiction. Conversely suppose every non-strong vertex of G is either a pendant vertex of G or is adjacent to a strong vertex of G. Suppose D_1 and D_2 are two minimal dsd sets of G. Let $u \in D_1 - D_2$. Then u cannot be pendant and u cannot be strong, since every dsd set contains all pendant and all strong vertices. Therefore $d(u) \geq 2$ and u is adjacent to a strong vertex v of G. Since D_1 is a minimal dsd set there exist v_1 in $V-D_1$ such that $N(v_1) \cap D_1 = \{u\}$ or there exist a v_2 in $V-D_1$ such that $N_s(v_2) \cap D_1 = \{u\}$ where $N_s(v_2) = \{x \in V / xv \in E(G) \text{ and } d(x) \geq d(v_2)\}$. Suppose there exist a v_1 in $V-D_1$ such that $N(v_1) \cap D_1 = \{u\}$. Therefore v_1 is neither a strong vertex nor a pendant vertex. Therefore there exist $w \in V(G)$ such that w is strong and v_1 is adjacent to w. But $w \in D_1$. So $N(v_1) \cap D_1 \supseteq \{u, w\}$, a contradiction. Suppose there exist a $v_2 \in V-D_1$ such that $N_s(v_2) \cap D_1 = \{u\}$. Since $v_2 \notin D_1$, v_2 is neither pendant nor strong. Arguing as before, we get a contradiction. So $D_1 - D_2 = \phi$. Hence $D_1 = D_2$. So G has a unique minimal dsd set. ■

3.1.31. Theorem:

For any graph G,

$\gamma_{dsd}(G)^{-1} + \gamma_t(\overline{G})^{-1} \leq 1$, where γ_t (G) demote the total domination number of G.

Proof:

Case(i): Suppose G has a full degree vertex.

Then $\gamma_t(G) = \infty$. $\gamma_{dsd}(G)^{-1} + \gamma_t(\overline{G})^{-1} \leq \frac{1}{2} + \frac{1}{\infty} = \frac{1}{2} < 1$.

Case (ii): Suppose G has no full degree vertex.

Then $\gamma_t(\overline{G}) \geq 2$. So $\gamma_{dsd}(G)^{-1} + \gamma_t(\overline{G})^{-1} \leq 1$. ■

3.1.32. Theorem :

$\gamma_{dsd}(G)^{-1} + \gamma_t(\overline{G})^{-1} = 1$ if and only if $G = \overline{K_2}$.

Proof:

Suppose $G = \overline{K_2}$. Then

$\gamma_{dsd}(G)^{-1} + \gamma_t(\overline{G})^{-1} = 1$. Conversely suppose $\gamma_{dsd}(G)^{-1} + \gamma_t(\overline{G})^{-1} = 1$. Therefore $\gamma_{dsd}(G) = 2 = \gamma_t(\overline{G})$. Suppose $|V(G)| \geq 3$. Since $\gamma_{dsd}(G) = 2$, there exists u,v such that $uv \notin E(G)$ and deg u = deg $v = \Delta = n-2 \geq 1$. So

$\overline{G}$ is disconnected containing {u,v} as a component. So $\gamma_t(\overline{G}) \geq 3$.

Hence $\gamma_{dsd}(G)^{-1} + \gamma_t(\overline{G})^{-1} \leq \frac{1}{2}+\frac{1}{3} < 1$, a contradiction. So $|V(G)| \leq 2$.

Clearly $|V(G)| \geq 2$. Hence $|V(G)| = 2$. Hence $G = \overline{K_2}$. ■

3.1.33. Theorem:

For any graph G, $\gamma_{dsd}(G)^{-1} + \gamma_t(\overline{G})^{-1} = \frac{1}{2}$.

Proof:

Since $\gamma_{dsd}(G) \geq 2$, $\gamma_{dsd}(G)^{-1} + \gamma_t(\overline{G})^{-1} = \frac{1}{2}$. if and only if $\gamma_{dsd}(G) = 2$ and $\gamma_t(\overline{G}) = \infty$. That is if and only if G has a full degree vertex and there exists u,v such that $uv \notin E(G)$ and deg u=deg v=n−2=Δ. Since G has a full degree vertex, Δ = n−1, a contradiction. ■

3.2. Excellent dom–strong domination:

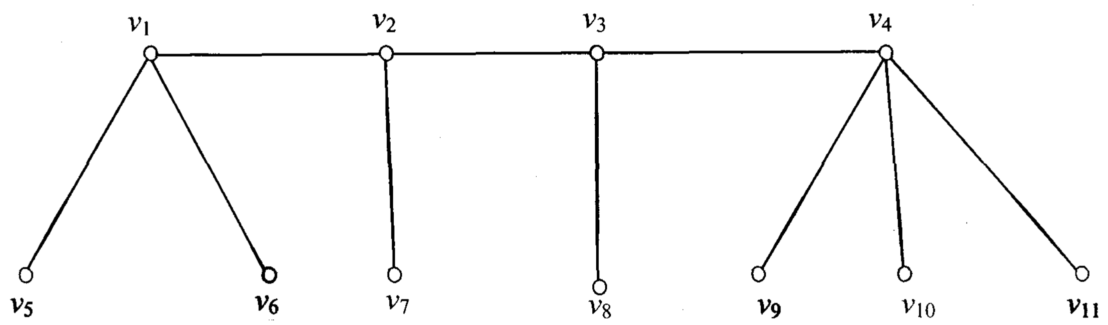
3.2.1. Definition:

Let G = (V, E) be a graph. A vertex $v \in V$ is called γ_{dsd} – good if v belongs to a γ_{dsd} – set.

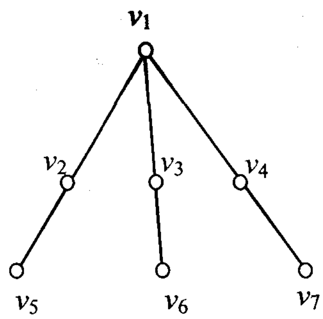
3.2.2. Definition:

A graph G is γ_{dsd} – excellent if every vertex v of V(G) is γ_{dsd} –good.

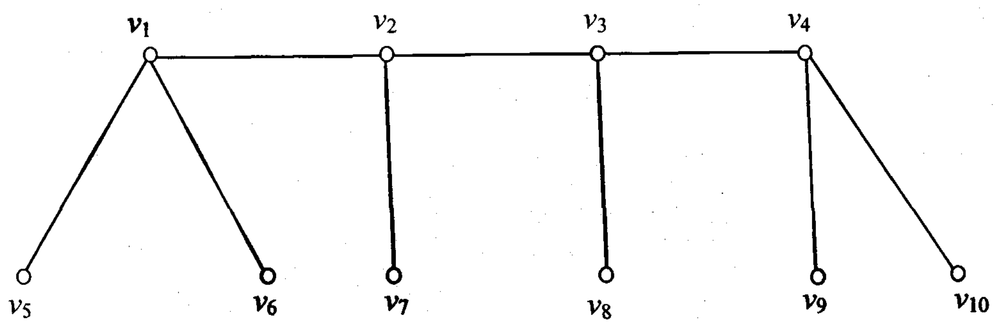
3.2.3. Examples:



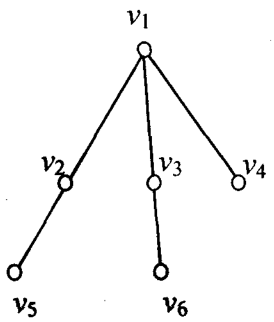
G_1



G_2



G_3



G_4

The graph G_1 is not γ_{dsd} –excellent since the vertex v_3 is γ_{dsd} – bad. The graph G_2 is also not γ_{dsd} – excellent since the vertices v_2, v_3 and v_4 are γ_{dsd} – bad, but it is γ_s – excellent. The graph G_4 is γ –excellent but neither γ_{dsd} – excellent nor γ_s – excellent. The graph G_3 is γ_{dsd} – excellent but neither γ – excellent nor γ_s – excellent.

3.2.4. Remark:

The carona $G \circ K_1$ is the graph of order 2n which is obtained from a copy of G by adding to each vertex $v \in V(G)$ a new vertex v' and a pendant edge uv'. Obviously G is an induced subgraph of $G \circ K_1$ and $\gamma(G \circ K_1) = \gamma_s(G \circ K_1) = n$. So $G \circ K_1$ is γ –excellent.

3.2.5. Proposition:

Every graph of order n is an induced subgraph of a γ_{dsd} –excellent graph and its cardinality is 2n.

Proof:

Let G be a graph of order n. To every vertex *v* attach a K_3. The resulting graph is a γ_{dsd} – excellent graph containing G as an induced subgraph and its γ_{dsd} is 2n.

3.2.6. Corollary:

There does not exist a forbidden subgraph, characterization of the class of γ_s –excellent graphs. For if there exists one, then we will get that G is not γ_s –excellent if it has a forbidden subgraph as induced subgraph. But then in this case there is a γ_s –excellent graph containing the forbidden subgraph as an induced subgraph.

3.2.7. Proposition:

The path P_n on n vertices is dsd – excellent if and only if $n \equiv 0,2,4 \pmod 6$.

Proof:

Let $P_n : u_1, u_2, u_3, \ldots u_n$ be a path on n vertices.

Case (i): Let $n \equiv 0 \pmod 3$.

Let $n = 3k$ with k even. Then $\{u_1,u_3,\ldots.u_{3k-1}, u_{3k}\}$ and $\{u_1,u_2,u_4\ldots.u_{3k-2}, u_{3k}\}$ are minimum dsd–excellent sets of cardinality $(3k+2)/2$. Hence P_n is dsd-excellent if $n=3k$ with k is even. If k is odd then $\{u_1,u_3,\ldots.u_{3k-2}, u_{3k}\}$ is the unique dsd set of cardinality $(3k+1)/2$. Hence P_n is not dsd–excellent if $n=3k$ with k is odd.

Case(ii): Let $n \equiv 1 \pmod 3$. Let $n = 3k+1$ with k is even. Then $\{u_1,u_3,\ldots.u_{3k-1}, u_{3k+1}\}$ is the unique minimum dsd–set of cardinality $(3k+2)/2$. Hence P_n is not dsd–excellent if $n=3k+1$ with k is even. If k is odd then $3k+1$ is even. In this case $\{u_1,u_3,\ldots.u_{3k}, u_{3k+1}\}$ and $\{u_1,u_2,u_4\ldots.u_{3k-1}, u_{3k+1}\}$ are minimum dsd–sets of cardinality $(3k+3)/2$. Hence P_n is dsd–excellent if $n=3k+1$ with k odd.

Case (iii): Let $n \equiv 2 \pmod 3$. Let $n = 3k+2$ with k is odd. Then $\{u_1,u_3,\ldots.u_{3k}, u_{3k+2}\}$ is the unique minimum dsd–set of cardinality $(3k+3)/2$. Hence P_n is not dsd–excellent if $n=3k+2$ with k is odd. If k is even then $3k+2$ is also even. In this case $\{u_1,u_3,\ldots.u_{3k+1}, u_{3k+2}\}$ and $\{u_1,u_2,u_4\ldots.u_{3k}, u_{3k+2}\}$ are minimum dsd–sets of cardinality $(3k+4)/2$. Hence P_n is dsd–excellent if $n=3k+2$ with k is even. ■

3.2.8. Proposition:

Let T be a tree with diam T≤ 5. Then T is dsd–excellent if and only if T is a caterpillar.

Proof:

Let T be a tree with diam T ≤ 5. Let $\{u_0, u_1,\ldots u_k\}$ be a set be a set of vertices in a longest path in T. If k =1. then T = K_2 and hence T is dsd–excellent. If k=2 then T is a star which is dsd–excellent. If k=3, then u_0, u_3 are pendant vertices and u_1, u_2 may have any number of pendant vertices. That is we have a double star with centres u_1, u_2 and a double star is clearly dsd–excellent. If k = 4 then u_0, u_4 are pendant vertices with every vertex in the neighborhood of u_1 or u_3 having degree 1. A vertex in the neighborhood of u_2 may have degree 1 or 2 if exactly one of deg u_1 or deg u_3 greater than deg u_2 and the other has degree less than or equal to deg u_2.

Case (i) :

u_2 has a path attached to it. In this case every dsd-set contains u_2. If deg u_1>deg u_2 and deg u_3>deg u_4 then T is γ_{dsd}-excellent, otherwise not.

Case (ii) :

u_2 has only pendant vertices in its neighborhood. If exactly one of deg u_1>deg u_2 or deg u_3>deg u_2 and the other less than or equal to deg u_2, then T is dsd-excellent, otherwise not.

If k=5:

Case (i) :

If u_2 and u_3 have paths attached to them then u_2 and u_3 are necessarily present in any dsd set. If deg u_1>degu_2 and deg u_4>deg u_3 then T is dsd-excellent, other wise not.

Case (ii):

Let u_2 have only a pendant vertex attached to it and u_3 has a path of length two attached to it (in case if u_3 has only a pendant vertex attached to it). If deg u_4>degu_3 and deg u_1=degu_2 then T is dsd-excellent, otherwise not.

Case (iii) :

Let u_2 and u_3 have only pendant vertices in their neighborhoods. In this case T is dsd-excellent if and only if deg u_1=deg u_3=deg u_4. Hence T is a double wounded spider with exactly one arm subdivided .

Case (iv) :

Let u_2 and u_3 have degree 2. Then also T is dsd-excellent if and only if deg u_1=deg u_2=deg u_3=deg u_4. ■

3.3. Split Dom-Strong Domination :

3.3.1. Definition :

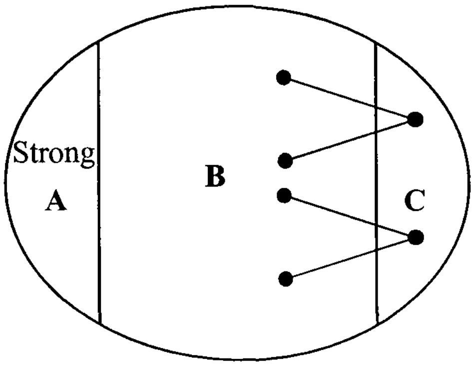
A dom-strong dominating set D of a graph G is a split dom-strong dominating set (sdsd-set) if V – D is disconnected. The cardinality of a minimum split dsd-set is denoted by $\gamma_{sdsd}(G)$.

3.3.2.Observation :

Let G be a non complete graph. Then there exists two points u,v which are not adjacent. If u and v are strong points (deg u>deg x, for all $x \in N(u)$ and deg v> deg y for all $y \in N(v)$) then G $-\{u,v\}$ cannot be complete. For if $G-\{u,v\}$ is complete then there exist a point x adjacent to u with deg x $\geq$ n-2. Hence deg u = n-1 this implies that u is adjacent to v, which is a contradiction, since u and v are non adjacent points. Therefore there exists $x,y \in G-\{u,v\}$ which are not adjacent.

3.3.3. Remark :

Assume that there are vertices which are not strong and the subgraph induced by the set of all non strong vertices is not complete and either contains a non complete component or contains atleast two non-trivial components. Then a split dsd set exists. If we assume that $\delta(G) \geq 2$ then a split dsd set exists.



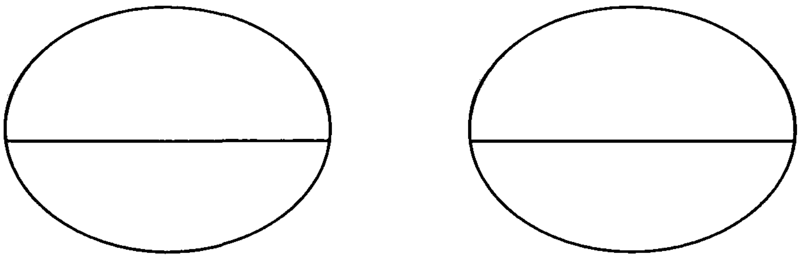
Non-strong

BUC=set of all non-strong vertices.

AUB is a split dsd set.

3.3.4. Remark :

Suppose G is a disconnected graph containing a minimum split dsd set. Then any minimum dsd set is a minimum split dsd set.



A B

AUB=V

Hereafter in this section we assume that G is a non-complete connected graph without strong isolates and $\delta(G) \geq 2$.

3.3.5. Observation :

If $\gamma_s(G)$ denotes the minimum split domination number of G then $\gamma_s(G) \le \gamma_{sdsd}(G)$. Likewise $\gamma_{ss}(G) \le \gamma_{sdsd}(G)$, where $\gamma_{ss}(G)$ denote the minimum split strong domination number of G.

3.3.6. Theorem :

$\gamma_{sdsd}(G) \le \alpha_o(G)$, where $\alpha_o(G)$ is the vertex covering number of G.

Proof :

Let S be a maximum independent set of vertices in G. If $|S|=1$ then G is complete, a contradiction. Therefore $|S| \ge 2$. Suppose there exists a vertex u in S which is not adjacent to any vertex in $V - S$ or adjacent to exactly one vertex in $V - S$ then deg $u \le 1$, a contradiction, since $\delta(G) \ge 2$. Therefore every vertex in S is adjacent to atleast two vertices in V–S. Since u is not a strong vertex and since $N(u) \subset V-S$, N(u) contains atleast two points, one of which strong dominates u. Therefore V–S is a split dsd set. Hence $\gamma_{dsd}(G) \le |V-S| = \alpha_o(G)$. ■

3.3.7. Theorem :

A dsd set D of G is a split dsd set if and only if there exists two vertices $w_1,w_2 \in V-D$ such every w_1-w_2 path contains a vertex of D.

Proof :

If there exists w_1,w_2 such that w_1-w_2 path contains a vertex of D then <V–D> is not connected. So D is a split dsd set. Conversely if D is a split dsd set then <V–D> is not connected. So take w_1,w_2 in different components of <V–D>. Then every w_1-w_2 path contains a vertex of D. ■

3.3.8 Theorem :

(i) $K(G) \leq \gamma_{sdsd}(G)$ where K is the connectivity of G.

(ii) $\gamma(G) \leq \gamma_s(G) \leq \gamma_{dsd}(G) \leq \gamma_{sdsd}(G)$.

Proof :

(i) follows from the fact that if D is a γ_{sdsd}-set of G then <V–D> is disconnected.

(ii) is obvious. ■

3.3.9 Theorem :

A split dsd set D is minimal if and only if for each vertex $v \in D$ one of the following holds:

(i) There exist a vertex $u \in V-D$ such that $N(u) \cap D = \{v\}$ or $N_s(u) \cap D = \{v\}$

(ii) v is a strong isolate in <D>

(iii) $((V-D) \cup \{v\})$ is connected.

Proof :

Suppose D is a minimal split dsd set such that v does not satisfy any of the above conditions. Then by (i) and (ii) $D-\{v\}$ is a dsd set, also since (iii) is not satisfied, $(V\text{-}D) \cup \{v\}$ is disconnected. Therefore $D-\{v\}$ is a split dsd set contradicting the minimality of D. Hence v satisfies one of the above conditions. Converse is obvious. ■

3.3.10. Remark :

Any superset S of a split dsd set D with $|S| \leq n-2$ is a split dsd set.

3.3.11. Theorem :

$$\gamma_{sdsd}(G) \leq \frac{n\Delta(G)}{\Delta(G)+1}$$

Proof :

Let D be a γ_{sdsd} set of G. Since D is minimal, for every vertex $v \in D$, $<(V-D) \cup \{v\}>$ is connected. Therefore there exist a vertex $u \in V-D$ such that v is adjacent to u. So V–D is a dominating set of G. Since $|V-D| \geq 2$, V–D is a dsd set of G.

Therefore $\gamma_{dsd}(G) \leq |V-D| \leq n - \gamma_{sdsd}(G)$.

But $\gamma_{dsd}(G) \geq \gamma(G) \geq \frac{n}{\Delta+1}$.

Hence $\gamma_{sdsd}(G) \leq n - \gamma_{dsd}(G)$

$$\leq n - \frac{n}{\Delta+1}$$

$$= \frac{n\Delta}{\Delta+1}$$

So $\gamma_{sdsd}(G) \leq \frac{n\Delta(G)}{\Delta(G)+1}$.

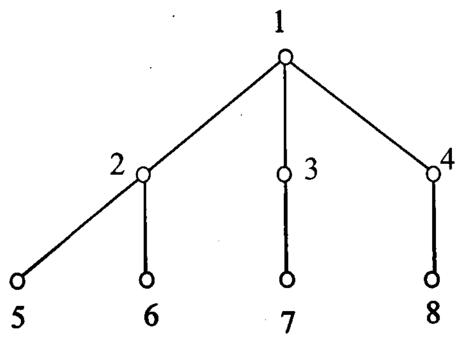
Hence the theorem. ∎

3.4. Dsd-domatic number $d_{dsd}(G)$:

3.4.1. Definition :

Let G=(V,E) be a simple graph. The dsd-domatic number d_{dsd} (G) of the graph G is defined as the maximum number of elements in a partition of V(G) into dom-strong dominating sets.

3.4.2 Example : Let G =



Here $d_{dsd}(G) = 1$

3.4.3. Definition :

Let G=(V,E) be a graph. Let $v \in V(G)$.

Let $N_s(v) = \{u \in v / uv \in E(G)$ and $\deg u \geq \deg v\}$ and

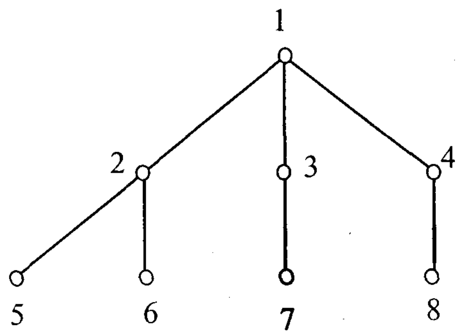
$N_s[v] = N_s(v) \cup \{v\}$, $d(v) = |N_s(v)|$ and $\delta_s(G) = \min\{d_s(v) / v \in V(G)\}$

3.4.4. Definition :

The strong domatic number of a graph G denoted by $d_s(G)$ is the maximum number of elements in a partition of V(G) into strong dominating sets.

3.4.5. Example :

Let G =



Here {2,3,4}and {1,5,6,7,8} are strong dominating sets and $d_s(G)$ =2, $\delta_s(G)$=1.

3.4.6. Proposition : $d_s(G) \leq \delta_s(G)+1$

Proof :

Let v be a point with $d_s(v)= \delta_s(v)$. Suppose $d_s(G)> \delta_s(G)+1$. Then there exist a strong dominating set of G which does not contain any of the elements of $N_s[v]$, a contradiction. Hence $d_s(G) \leq \delta_s(G) +1$. ■

3.4.7. Remark : The domatic partition into dsd sets consists of V only, since the pendant vertices must be in every dsd-set of G.

3.4.8. Remark :

If G has a pendant vertex then $d_{dsd}(G)$=1. Hence for a tree T, $d_{dsd}(T) = 1$.

3.4.9. Definition :

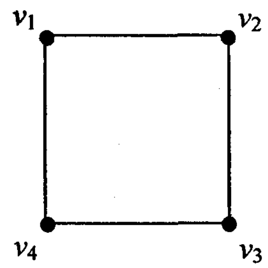
Let $v \in V(G)$, v is said to be ds-isolate if $d(v) \leq 1$ or $d(v) \geq 2$ and $d_s(v)=0$.

3.4.10. Remark :

For any graph G, $d_{dsd}(G) \leq \min\{\delta(G)+1, \delta_s(G)+1\}$

3.4.11. Observation : $1 \leq d_{dsd}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$

Consider C_4 :



Here $\{v_1, v_3\}$ and $\{v_2, v_4\}$ are the only dsd sets and

$V(C_4) = \{v_1, v_3\} \cup \{v_2, v_4\}$ and $d_{dsd}(C_4)=2$.

For K_n : $d_{dsd}(K_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

3.4.12. Theorem :

d_{dsd} (G) + $\overline{d_{dsd}}$ (G) ≤n+1, where $\overline{d_{dsd}}$ denote dsd-domatic number of $\overline{G}$.

Proof :

Since any dsd-partition of G is a domatic partition of G, we have d_{dsd} (G) ≤ d(G). Therefore d_{dsd}(G)+ $\overline{d_{dsd}}$ (G) ≤ d(G)+ $\overline{d}(G)$ ≤ n+1, where $\overline{d}(G)$ denote the domatic partition on $\overline{G}$. ■

3.4.13. Lemma :

Let P={D_1,D_2, ... D_k} be a dsd-partition of V. Then k ≤ $\left\lfloor \frac{n}{2} \right\rfloor$.

Proof : Since |D_i| ≥2, for each *i*, we get k ≤ $\left\lfloor \frac{n}{2} \right\rfloor$. ■

3.4.14. Theorem :

Let G be a graph with a strong isolated point or a pendant vertex.

Then d_{dsd} (G)+ $\overline{d_{dsd}}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$

Proof :

Since G has a strong vertex or a pendant vertex d_{dsd}(G)=1.

By Lemma 3.4.13, $\overline{d_{dsd}}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Therefore d_{dsd} (G)+ $\overline{d_{dsd}}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$ + 1. ■

3.4.15. Remark :

If G=K_n or $\overline{K_n}$ then $d_{dsd}(G)+\overline{d_{dsd}}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$

Proof :

If G = K_n or $\overline{K_n}$ then $d_{dsd}(K_n) = \left\lfloor \frac{n}{2} \right\rfloor$ and $\overline{d_{dsd}}(K_n) = 1$.

Hence $d_{dsd}(K_n)+\overline{d_{dsd}}(K_n) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$ ■

3.4.16. Remark :

If G = $K_4 - \{e\}$ then $d_{dsd}(G)+\overline{d_{dsd}}(G) = \left\lfloor \frac{n}{2} \right\rfloor + 1$. The converse of the result in the Remark 3.4.15 is not true.

3.4.17. Proposition :

For any graph G, $d_{dsd}(G) \leq \min\{\delta(G)+1, \ \delta_s(G)+1\}$,

(Remark: 3.4.10)

Proof :

Since every dsd set is a dominating set as well as a strong dominating set we get that every dsd-domatic partition is both a domatic partition and a strong domatic partition and hence $d_{dsd}(G) \leq d(G)$. But $d(G) \leq \delta(G)+1$ and $d_s(G) \leq \delta_s(G)+1$.

Therefore $d_{dsd}(G) \leq \min\{\delta(G)+1, \ \delta_s(G)+1\}$. ■

3.4.18. Remark :

Let G be a graph with $\delta(G) \geq 1$ and $\Delta(G) \leq n-2$.

Then $d_{dsd}(G) + \overline{d_{dsd}}(G) \leq n+1$.

Proof :

We have $d_{dsd}(G) \leq \delta(G)+1$ and $\overline{d_{dsd}}(G) \leq \overline{\delta}(G)+1 \leq \overline{\Delta}(G)+1$. So

$d_{dsd}(G) + \overline{d_{dsd}}(G) \leq \delta(G) + \overline{\Delta}(G) + 2$

$= n-1+2 = n+1$ (Since $\delta(G) + \overline{\Delta}(G) = n-1$)

Where $\overline{\delta}$ and $\overline{\Delta}$ represents the minimum and maximum degree respectively on $\overline{G}$. ■

3.4.19. Definition :

A graph G is

(i) strong domatically full if $d_s(G) = \delta_s(G)+1$

(ii) dsd-domatically full if either $\delta(G)=1$ or $\delta(G) \geq 2$ and

$d_{dsd}(G) = \min \{\delta(G)+1, \delta_s(G)+1\}$.

3.4.20. Proposition :

For any graph G,

(i) $d_s(G) \le \frac{n}{\gamma_s(G)}, d_s(G) \ge \left\lfloor \frac{n}{n-\delta_s(G)} \right\rfloor$

(ii) $d_{dsd}(G) \le \frac{n}{\gamma_{dsd}(G)}, d_{dsd}(G) \ge \max\left\{\frac{n}{n-\delta(G)}, \frac{n}{n-\delta_s(G)}\right\}$.

Proof :

Obviously $d_s(G) \le \frac{n}{\gamma_s(G)}$ and $d_{dsd}(G) \le \frac{n}{\gamma_{dsd}(G)}$. Let $S \subseteq V$ be such that $|S| \ge n-\delta_s(G)$. Let $v \in V-S$. Since $|N_s(v)| \ge 1+\delta_s(G)$, we have $N_s(v) \cap S \ne \phi$. Therefore any set of cardinality greater than or equal to $n-\delta_s(G)$ is a strong dominating set. Therefore we can take any $\left\lfloor \frac{n}{n-\delta_s(G)} \right\rfloor$ disjoint subsets and each of these is a strong dominating set. Therefore $d_s(G) \ge \left\lfloor \frac{n}{n-\delta_s(G)} \right\rfloor$. Also since any ds-domatic partition is both a domatic partition and a strong domatic partition, we have $d_{dsd}(G) \ge \max\left\{\frac{n}{n-\delta(G)}, \frac{n}{n-\delta_s(G)}\right\}$. ■

3.4.21. Result :

P_n is strong-domatically full for all $n \geq 1$.

Proof :

Let $n \geq 4$, Let $V(P_n) = \{u_1, u_2, \ldots\ldots u_n\}$. Then $\{u_1, u_3 \ldots\ldots\}$ and $\{u_2, u_4, \ldots\ldots\}$ are strong dominating sets, since $\delta_s(P_n)=1$. Therefore P_n is strong domatically full.

When $n=3$, $\delta_s(P_3)=0$ and P_3 is the only domatic partition in the partition containing V. Therefore P_3 is strong domatically full. Obviously P_1, P_2 are strong domatically full. ■

3.4.22. Remark :

P_n is dsd-domatically full for all $n \geq 1$.

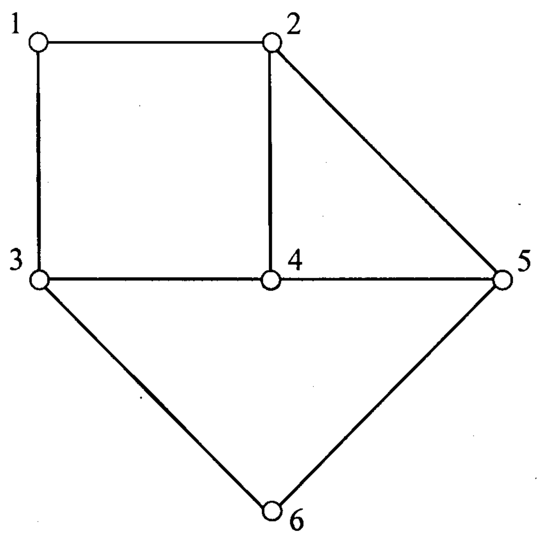
3.4.23. Definition :

A graph G is said to be strong domatically k-critical if $d_s(G)=k \geq 2$ and $d_s(G-e) < k$, for every edge e.

3.4.24. Remark :

If G has a strong isolated point, then d_s (G) = 1

For :



G

G has no strong isolates. D = {4,3} is strong dominating set of G, but V–D = {1,2,5,6} is not a strong dominating set of G.

Also D_1 = {3,5} is a strong dominating set and V–D_1 = {1,2,4,6} is also a strong dominating set of G.

3.4.25. Result : Let G be strong - domatic 2- critical. Then G is strong domatically full.

Proof :

By hypothesis $d_s(G) = 2$ and $d_s(G\text{-}e)=1$, for every edge $e \in E(G)$. That is G has no strong - isolated point and G-e has a strong isolated point, for every $e \in E(G)$. Let $e=uv$. Then removal of e generates a strong isolated point. Therefore either u or v is a strong isolated point of G-e. Therefore for every point $w \in (N(u)-\{v\})$, deg $w \leq$ deg $u - 2$ or for every point $w \in (N(v)-\{u\})$, deg $w \leq$ deg $v - 2$. So $1+\delta_s(G) = 2$ and $d_s(G) = 2$. Hence $\delta_s(u)=1$ or $\delta_s(v)=1$. So $d_s(G)=1+\delta_s(G)$. Hence G is strong domatically full. ■

♦ ♦ ♦

CHAPTER - IV

In this chapter we introduce independent dsd, connected dsd and total dsd concepts among graphs. dsd-irredundance is also discussed.

4.1. Independent dom-strong domination :

4.1.1. Definition :

Let G=(V,E) be a graph. A dsd set D is called an independent dsd set if <D> is totally disconnected.

4.1.2. Definition :

Let G be a graph which contains an independent dsd set. Then define

$i_{dsd}(G)$ =min {|D|:D is a minimal independent dsd set} and

$\beta_{dsd}(G)$=max{|D| : D is a minimal independent dsd set}

4.1.3. Definition :

Let $v \in V$. Define

$$i^s_{dsd}(v) = \begin{cases} \min\{|D| : D \text{ is a minimal independent dsd set containing } v\} \\ 0 \text{ if there exist no independent dsd set containing } v \end{cases}$$

and $i^s_{dsd}(G) = \max\limits_{v \in V} \{i^s_{dsd}(v)\}$

4.1.4. Remark :

Not all graphs have independent dsd sets.

(i) K_n has no independent dsd set.

(ii) C_{2n} $(n \geq 2)$ have independent dsd set, C_{2n+1} $(n \geq 1)$ has no independent dsd set.

(iii) P_{2n+1} $(n \geq 2)$ have independent dsd sets, but P_{2n} $(n \geq 1)$ does not have independent dsd set.

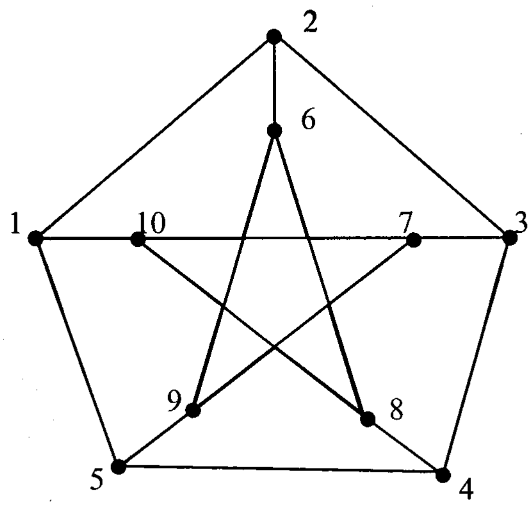
(iv) W_n has no independent dsd set for all $n \geq 4$.

(v) $K_{1,n}$ has no independent dsd set.

(vi) $K_{m,n}$ has independent dsd sets provided $m,n \geq 2$. If $m>n$ then there exist a unique independent dsd set namely the set of n vertices forming a partition. If $m=n$ then there exist two independent dsd sets namely the two partitions.

(vii) The Petersen graph has independent dsd sets.

For ex : P :



Here {2,4,9,10} and {1,3,8,9} are independent dsd sets.

Hence $i_{dsd}(P) = \beta_{dsd}(P) = 4$.

(viii) F_n has no independent dsd set.

4.1.5. Theorem :

If G has two strong isolated vertices which are adjacent then G has no independent dsd set.

Proof :

Any dsd set contains all strong isolated vertices. Hence the theorem. ■

4.1.6. Remark:

If $v \in V(G)$ is such that every dsd set contains $N[v]$ then G has no independent dsd set.

4.1.7. Theorem :

If G has a full degree vertex then G has no independent dsd set.

Proof :

Let u be full degree vertex in G. Suppose there exist an independent dsd set say D. Then $|D| \geq 2$ and $u \notin D$. (If $u \in D$ then u being a full degree vertex is adjacent to every point of $D-\{u\} \neq \phi$, contradicting the independence of D.) Since D is a dsd set, there exist a $v \in D$ which strongly dominates u. Hence v is a full degree vertex in D, a contradiction to the independence of D. Hence G has no independent dsd set. ■

4.1.8. Remark :

For any graph G, G^+ (Carona) does not have any independent dsd set.

Proof :

Let D be any dsd set. Let $V(G)= \{u_1, u_2 \ldots\ldots u_n\}$. Let v_i be the pendant vertex of G^+ adjacent to u_i. $(1 \leq i \leq n)$. Then $\{v_1, v_2 \ldots\ldots v_n\} \subset D$. For u_i there exist $u_j \in D$ such that u_j is adjacent to u_i. Therefore there exist an edge u_iu_j in <D>. So D is not an independent dsd set.

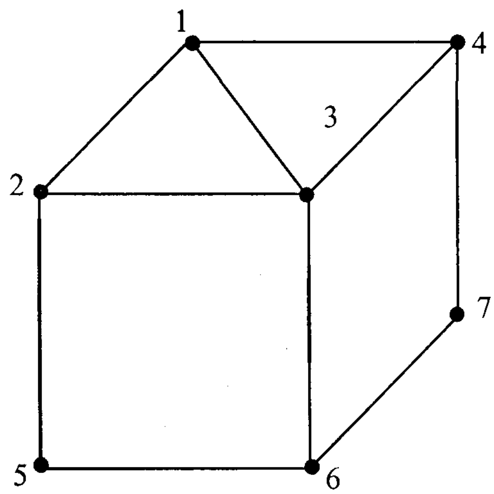
Hence the proof. ■

4.1.9. Remark :

Let G=(V,E) be a simple graph. If $u \in V(G)$ is a support and every vertex in N(u) is either a pendant or a support then G does not have any independent dsd set.

4.1.10 Example :

Consider the graph,



$3 \in V(G)$ is a strong isolate and $N(1) \subseteq N(3)$. So G has no independent dsd set.

4.1.11. Result :

Let G=(V,E) be a simple graph. If $u \in V(G)$ is a strong isolate and if there exist $v \in N(u)$ such that $N(v) \subseteq N(u)$ then G has no independent dsd set.

Proof :

Since u is a strong isolate, any dsd set D contains u. For dom-strong domination of v, D must contain a neighbor of v other than u. since $N(v) \subseteq N(u)$, <D> contains an edge. Hence the result. ■

4.1.12. Result :

Let G=(V,E) be a simple graph. Any independent dsd set contains link-complete vertices.

Proof :

Let D be an independent dsd set. Let $u \in V(G)$ be a link-complete vertex. Suppose $u \notin D$. Then the two points in D which dominate u will be adjacent, a contradiction. ■

4.1.13. Result :

Let G=(V,E) be a simple graph. Let $u \in V(G)$. Let either N[u] or $N_s[u]$ be contained in $N(v)$, where v is a link complete or a strong isolate vertex. Then G has no independent dsd set.

Proof :

Let D be any dsd set. Then $v \in D$. Either $v \in D$ or two neighbors of u, one of which being strong belong to D. Hence D contains an edge. ■

4.1.14. Corollary :

If G has at least two adjacent link-complete vertices then G has no independent dsd set.

4.1.15. Result :

Any cycle with atleast four vertices and a chord does not have any independent dsd set.

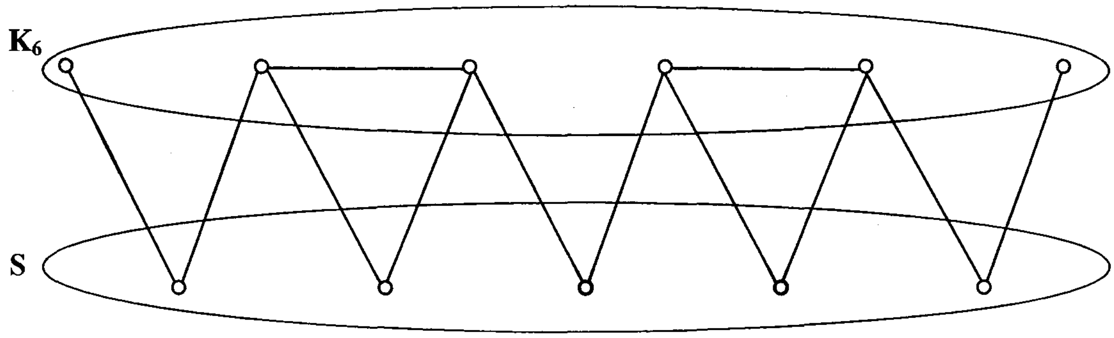
Proof :

Let uv be a chord in a cycle C_n ($n \geq 4$). Then u and v have degree 3. Let D be an independent dsd set. Either u or v belong to D but not both. Suppose $u \in D$. Then $v \notin D$. Let w be a point adjacent to v. Then deg w=2. Since $v \notin D$, there is no point in D strongly dominating w, a contradiction.

■

4.1.16 Result :

1. A graph G in which there exists y_1, y_2 such that deg y_1= deg $y_2 = \Delta = n-2$ has an independent dsd set.
2. Let $S \subseteq V$. Let $S = \{u_1, u_2, --- u_r\}$, $d_i \leq d_G(u_i, S-\{u_i\})$. Then $(d_1, d_2, ... d_r)$ is a distance sequence of S. If S is maximal independent then $2 \leq d_i \leq 3$, for every i.
3. A graph G has an independent double dominating set if and only if there exist a maximal independent set S for which the distance sequence of S is 2,2,2, ...



The above graph satisfies the condition of the hypothesis. (It has a independent dd-set but no independent dsd-set)

4.2. Dsd-irredundance in graphs :

4.2.1. Definition :

Let G=(V,E) be a graph. A subset D of V is called a dsd-irredundent set if for every $u \in D$ one of the following holds :

i) u is a pendant vertex of G

ii) u is an isolate in <D> or a strong isolate in <D> or $|N(u) \cap D|=1$.

iii) There exist $v_1 \in V\text{-}D$ such that $N(v_1) \cap D=\{u\}$ or there exists a $v_2 \in V\text{-}D$ such that $N_s(v_2) \cap D=\{u\}$.

iv) There exist a $v \in V-D$ such that $u \in N(v)$ and $|N(v) \cap D|=2$.

4.2.2. Definition :

The maximum cardinality of a maximal dsd-irredundant set is called upper dsd-irredundant number and is denoted by IR_{dsd} (G). Also the minimum cardinality of a maximal dsd-irredundant set is called lower dsd-irredundant number and is denoted by ir_{dsd} (G).

4.2.3. Definition :

$ir^{s}_{dsd}(v)$ = min {|D| : D is a maximal dsd-irredundant set containing v}

$$ir^{s}_{dsd}(G) = \max_{v \in V} \{ir^{s}_{dsd}(v)\}$$

Note that the dsd-irredundant property is super hereditary.

4.2.4. Recall :

Let G = (V, E) be a graph. Let D be a subset of V. Let $v \in V$. Define,

$pn[v,D] = N[v] - N[D-\{v\}]$ and

$pn_s[v,D] = N_w[v] - N_w[D-\{v\}]$.

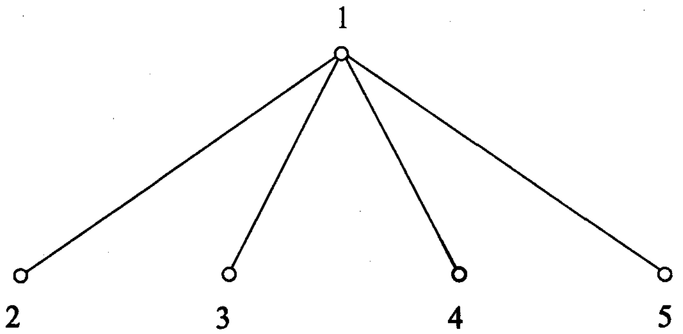
4.2.5. Remark :

Suppose we define the dsd-irredundant set as follows :

$D \subset V$ is dsd-irredundant if for every $u \in D$, $pn[u,D] \neq \phi$ or $pn_s[u,D] \neq \phi$. Then a minimum dsd set need not be a dsd-irredundant set.

4.2.6. Example:

Consider the following star :



Here D={1,2,3,4,5} is the minimum dsd-set.

pn [1,D] = N [1] – N[D–{1}] =ϕ

$pn_s[1,D]$= {1}

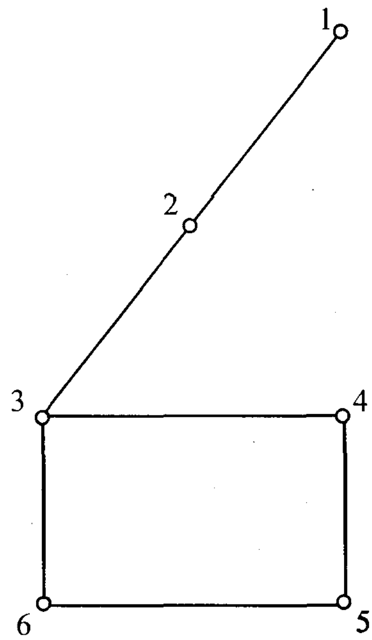
pn[2,D]=ϕ

$pn_s[2,D]=\phi$

So D is not a dsd-irredundant set.

4.2.7. Example :

Consider the graph,



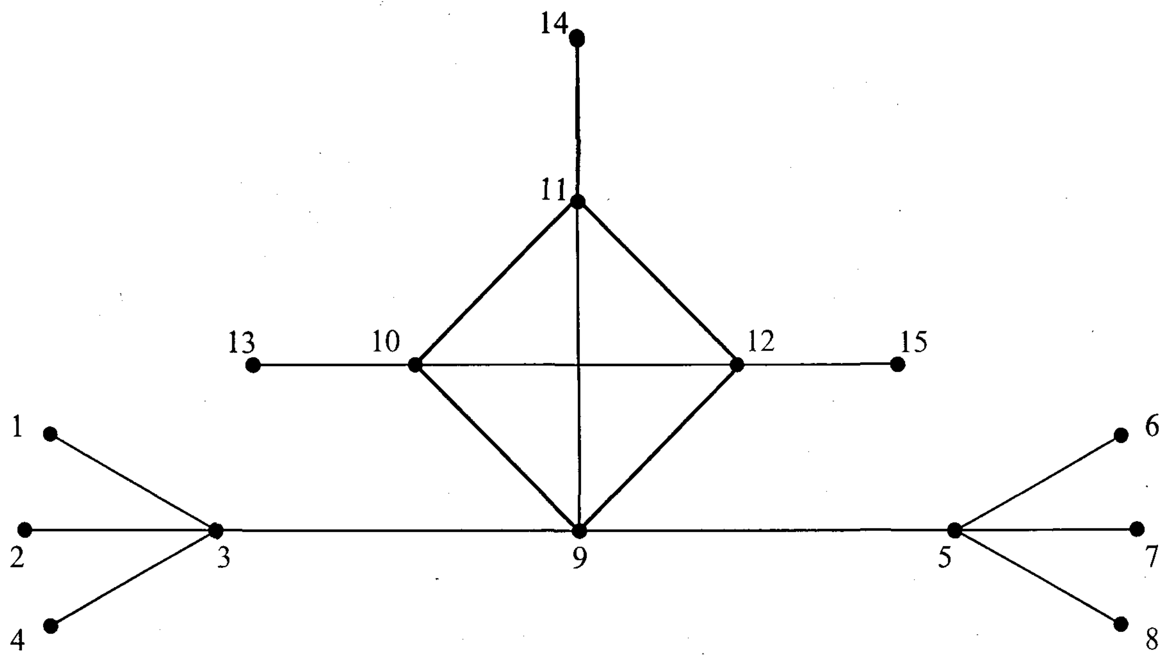
Here D_1 = {1,3,5} is an independent dsd-set. Any superset of D_1, is not independent. D_2={1,3} is independent but not a dsd-set. So any subset of D_1 is not a dsd set but it is independent. D_3= {1,4,6} is an independent dsd-set and no superset or subset of D_3 is a independent dsd-set. D_1 and D_3 are the only minimal independent dsd sets.

Also $i^s_{dsd}(2) = 0, i^s_{dsd}(1) = i^s_{dsd}(3) = i^s_{dsd}(4) = i^s_{dsd}(5) = i^s_{dsd}(6) = 3$

So $i^s_{dsd}(G) = 3$.

{1,4,6}, {2,5,6} and {1,3,5} are all dsd-irredundant sets. It shows that no four element set is irredundant. Hence $ir^s_{dsd}(G) = 3$

4.2.8. Example :



D_1= {3,5,9,10,11,12} is a maximal irredundant set.

D_2 = {1,2,4,6,7,8,9,13,14,15} is a maximal irredundant set. Similarly

D_3={1,2,4,5,9,13,14,15},

D_4={3,6,7,8,9,13,14,15}

D_5={1,2,4,5,9,10,11,12} and

D_6 ={3,5,13,14,15,9} are all maximal irredundant sets. So $ir_{dsd}(G)$ =6,

$IR_{dsd}(G)$=10, $ir^s_{dsd}(G)=8$. The set D_2={1,2,4,6,7,8,9,13,14,15} is the only

minimal independent dsd set. So $i_{dsd}(G)=\beta_{dsd}(G)=10$ and $ir^s_{dsd}(G)=10$.

Since D_2 is the unique minimal dsd set, $\gamma_{dsd}(G)=10=\Gamma_{dsd}(G)$.

4.2.9. Theorem :

For any graph G, ir $(G) \leq ir_{dsd}$ (G)

Proof :

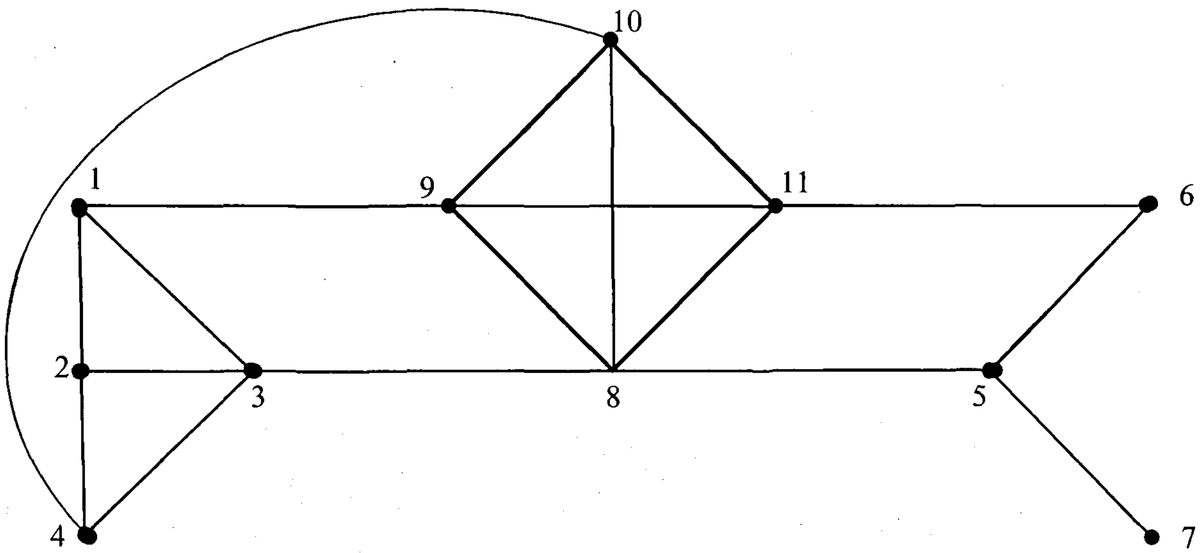
Let D be a maximal irredundant set of G. Then for every $u \in D$, $pn[u,D] \neq \phi$. So D is a dsd-irredundant set. Therefore there exist a subset T of V such that $D \subset T$ and T is a maximal dsd-irredundant set. Hence

min {|D| : D is a maximal dsd-irredundant set }

≤min {|T| : T is a maximal dsd-irredundant set }

Hence ir (G) ≤ $ir_{dsd}(G)$. ■

4.2.10. Example : Consider the graph,



Here D_1= {1,4,6,7,8 } is an independent dsd-set.

D_2 = {2,3,5,8,10,11} and D_3 = {1,3,4,6,78 } are maximal irredundant sets.

So $i_{dsd} = \beta_{dsd} = i^s_{dsd} = \gamma_{dsd} = ir_{dsd} = 5$ and $\Gamma_{dsd} = ir^s_{dsd} = IR_{dsd} = 6$

4.2.11. Theorem :

Every minimal dsd set is maximal dsd-irredundant.

Proof :

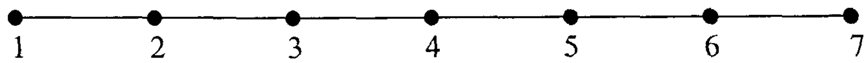
Every minimal dsd set is clearly dsd-irredundant. Let D be a minimal dsd set. Suppose D is not a maximal irredundant set. Then there exist a vertex u∈V-D such that D∪{u} is a dsd-irredundant set. Let S=D∪{u}. Since u∈V-D and D is a dsd set, u cannot be a pendant vertex of G. If u is an isolate of <S> or a strong isolate of <S> or |N(u)∩S|=1 then D cannot dom-strong dominate u, a contradiction. If v_1, v_2∈V−S such that N(v_1)∩S={u} or $N_s(v_2)$ ∩S={u}. Hence N(v_1) ∩D=ϕ or $N_s(v_2)$ ∩D=ϕ, a contradiction to the fact that D is a dsd set. If u satisfies (iv) then for some v∈V−D such that u∈N(v), |N(v)∩S|=2. So

$|N(v) \cap D|=1$. Hence D does not dom-strong dominate v, a contradiction, since D is a dsd-set. Hence D is a maximal irredundant set. ■

4.2.12. Remark :

There exist a graph G for which $ir^{s}_{dsd}(G) > \gamma_{dsd}(G)$.

Consider G=P_7 :



Here {1,2,4,6,7} is the only minimal dsd set containing 2. So $ir^{s}_{dsd}(P_7) = 5$

{1,3,5,7} is the only minimum dsd set. So $\gamma_{dsd}(G)=4$.

Also $ir^{s}_{dsd}(1) = ir^{s}_{dsd}(3) = ir^{s}_{dsd}(5) = ir^{s}_{dsd}(7) = 4$.

So $ir^{s}_{dsd}(P_7) = 5 = \gamma_{dsd}(P_7) + 1 > \gamma_{dsd}(P_7)$.

4.2.13. Remark :

A graph G is dsd-excellent is every point lies in a γ_{dsd}- set. If G is dsd-excellent then $ir^{s}_{dsd}(G) \le \gamma_{dsd}(G)$.

4.2.14. Theorem :

For any graph G, $\frac{\gamma_{dsd}(G)}{2} < ir_{dsd}(G) \le \gamma_{dsd}(G) \le 2ir_{dsd}(G) - 1$

Proof :

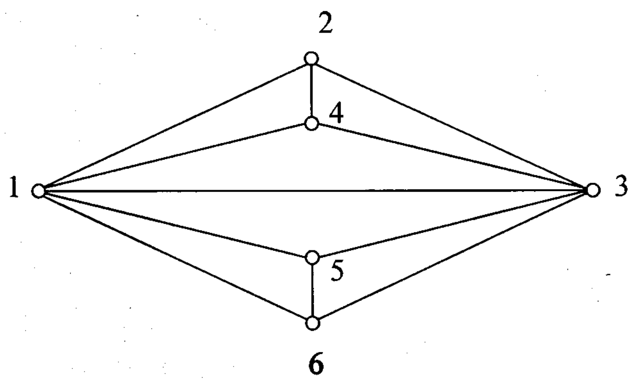
Let $ir_{dsd}(G)$ =k. Let S={$v_1,v_2,......v_k$} be a ir_{dsd}-set of G. If v_i is neither a pendant in G nor an isolate in <S> nor a strong isolate in <S> then there exist $u_i \in V$-S such that $N(u_i) \cap S=\{u_i\}$ or $N_s(u_i) \cap S=\{v_i\}$ or $N(u_i) \cap S=\{v_i, v_j\}$. Let S' = {$u_1, u_2,.......u_k$} where $u_i=v_i$ if v_i is a pendant in G or an isolate in <S> or a strong isolate in <S>. Suppose S"=S∪S' is not a dsd set. Then there exist w∈V-S" such that w is not dom-strong dominated by S". w may be a pendant vertex of G. Either $w \notin N(x)$ for any $x \in$S" or $x \in N(w)$ for some $x \in$S" but there exist no $x \in$S" such that $x \in N_s(w)$ or $x \in N_s(w)$ for some $x \in$S" and there exist no y∈S", $y \neq x$ which is adjacent to w. Consider S∪{w}. If $w \notin N(x)$ for any $x \notin$ S" then w is an isolate of <S"∪{w}> and hence an isolate of <S∪ {w}>. If there exist no $x \in$S" such that $x \in N_s(w)$ then w is a strong isolate of <S"∪ {w}> and hence a strong isolate of <S∪ {w}>. If $x \in N_s(w)$ for some $x \in$S" and there exist no y∈S", $y \neq x$ which is adjacent to w then $N_s(w) \cap S"=\{x\}$. If $x \in$S then $N_s(w) \cap S"=\{x\}$, w is a strong private neighbor of $x=v_i$ in S. Therefore w∈S', a contradiction. If $x \notin$S then $x=u_i$ for some $u_i \in$S'. Then $u_i \neq v_i$ and $N(u_i) \cap S=\{v_i\}$ or $N_s(u_i) \cap S=\{v_i\}$ or $N(u_i) \cap S=\{v_i,v_j\}$. Therefore $N_s(w) \cap (S \cup \{u_i\})=\{u_i\}$. So either S∪{w} or S∪{$u_i$} is a dsd-irredundant set, a contradiction, since S is a maximal dsd-irredundant set. Therefore S" is a dsd set. It cannot be a minimal dsd set. For: If S" is a minimal dsd set then S" is a maximal dsd-irredundant set containing S which is maximal. So S"= S. Hence S'=ϕ. Then every element v of S is a pendant in G or an isolate in <S> or a strong isolate of <S>. So S"=S. Hence S=ϕ, a contradiction. Hence S" is not a minimal dsd set. Therefore $\gamma_{dsd}(G)<|S''|=2k$.

Therefore $\gamma_{dsd}(G)\leq 2k-1=2ir_{dsd}(G)-1$.

So $\gamma_{dsd}(G)<2ir_{dsd}(G)$. Hence $\frac{\gamma_{dsd}(G)}{2}<ir_{dsd}(G)$. ■

4.2.15. Example :

Consider the graph,



Here {1} is a maximal dsd-irredundant set. {1,3} is a minimum dsd set. So $ir_{dsd}=1$ and $i_{dsd}=\beta_{dsd}=0$. Therefore there exist no independent dsd set.

4.2.16. Remark :

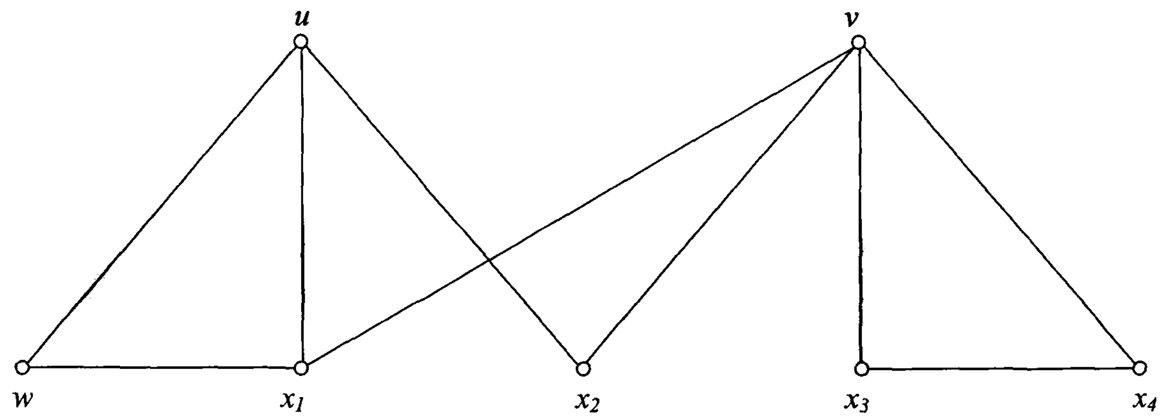
If $ir_{dsd}(G)=1$ then there exist a full degree vertex and hence there exist no independent dsd set. So $i_{dsd}(G)$ is ∞.

4.2.17. Remark :

If $ir(G)=1$ then $i(G)=\gamma(G)=1$. But if $ir_{dsd}(G)=1$ then $i_{dsd}(G)$ is ∞ and hence $ir_{dsd}(G) \neq i_{dsd}(G)$.

4.2.18. Example :

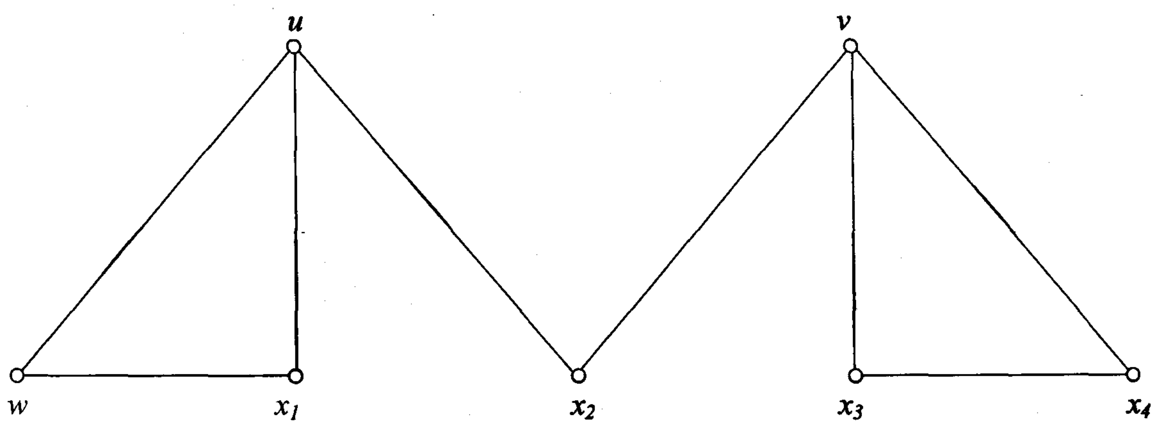
Consider the graph,



Here $D=\{u,v,x_1,x_3\}$ is a minimum dsd set. $|N(u)\cap D|=|\{x_1\}|=1$. For $x_4 \in V-D$, $x_4 \in N(v)$ and $|N(x_4)\cap D|=2$. Similarly for $x_2 \in V-D$, $x_2 \in N(v)$ and $|N(x_2)\cap D|=2$. For $w \in V-D$, $w \in N(x)$ and $|N(w)\cap D| = 2$.

Also $|N(x_3) \cap D|=1$. Hence u and x_3 satisfy condition (ii), v and x_1 do not satisfy (i), (ii) and (iii) but satisfy (iv).

4.2.19. Example : Consider



Here D= $\{u, v, x_1, x_3\}$ is a minimum dsd set.

<D> =

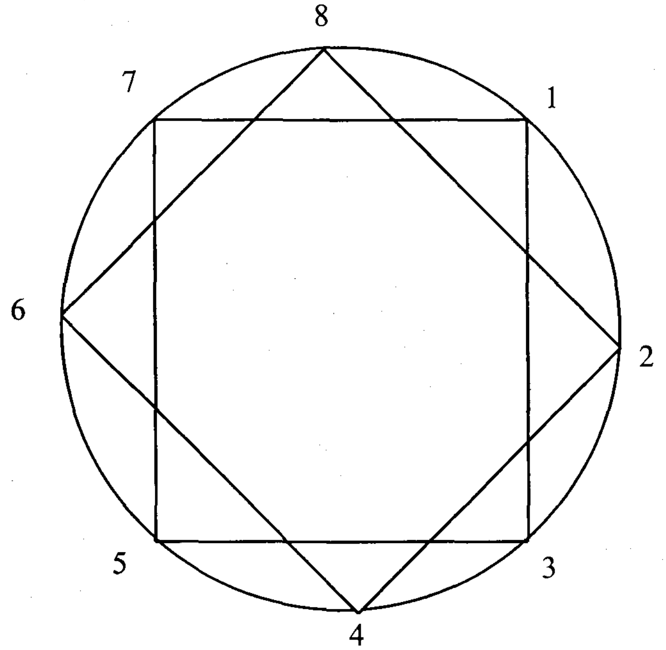
None of the elements of D are pendant vertices. None of them are isolates or strong isolates of <D>. $|N(u) \cap D|=1$, $|N(v) \cap D|=1$, $|N(x_1) \cap D|=1$ and

$|N(x_3)\cap D|=1$ $N(w)\cap D=\{u,x_1\}=N_s(w)\cap D$ and $N(x_2)\cap D=\{u,v\}=N_s(x_2)\cap D$. So u has no private neighbor and hence no strong private neighbor in V−D. Also u satisfies condition (ii) and (iv) $N(x_4)= \{v,x_3\}=N_s(x_4)$. So v, x_1, x_3 do not have private neighbor nor strong private neighbor in V−D. They all satisfy condition (iv). All the vertices in D satisfy conditions (ii) and (iv).

4.2.20.Remark :

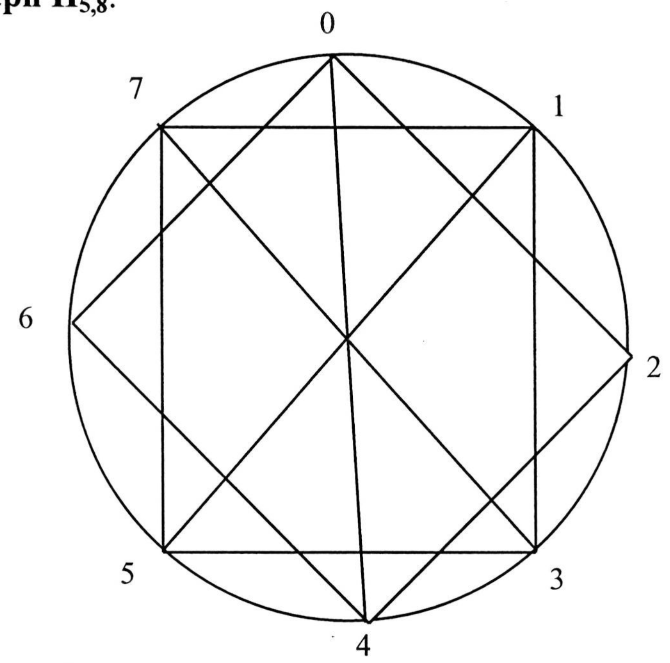
We establish these domination parameters to some **Familiar graphs**:

1. Harary graph $H_{4,8}$:



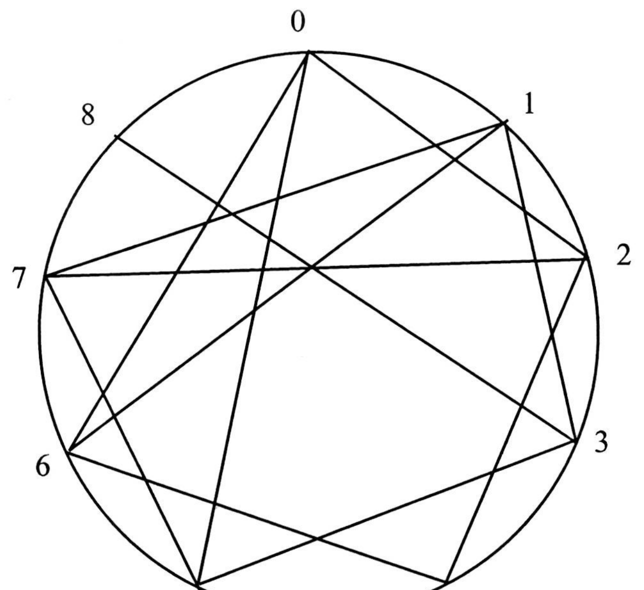
$D_1= \{1,5,4,8\}$, $D_2=\{2,3,6,7\}$ are minimum dsd sets which are not independent. Also $D_3= \{2,4,6,8\}$, $D_4=\{1,3,5,7\}$ are maximal dsd-irredundant sets.

2. Harary graph $H_{5,8}$:



$D_1=\{0,2,4,6\}$ and $D_2=\{1,3,5,7\}$ are dsd sets.

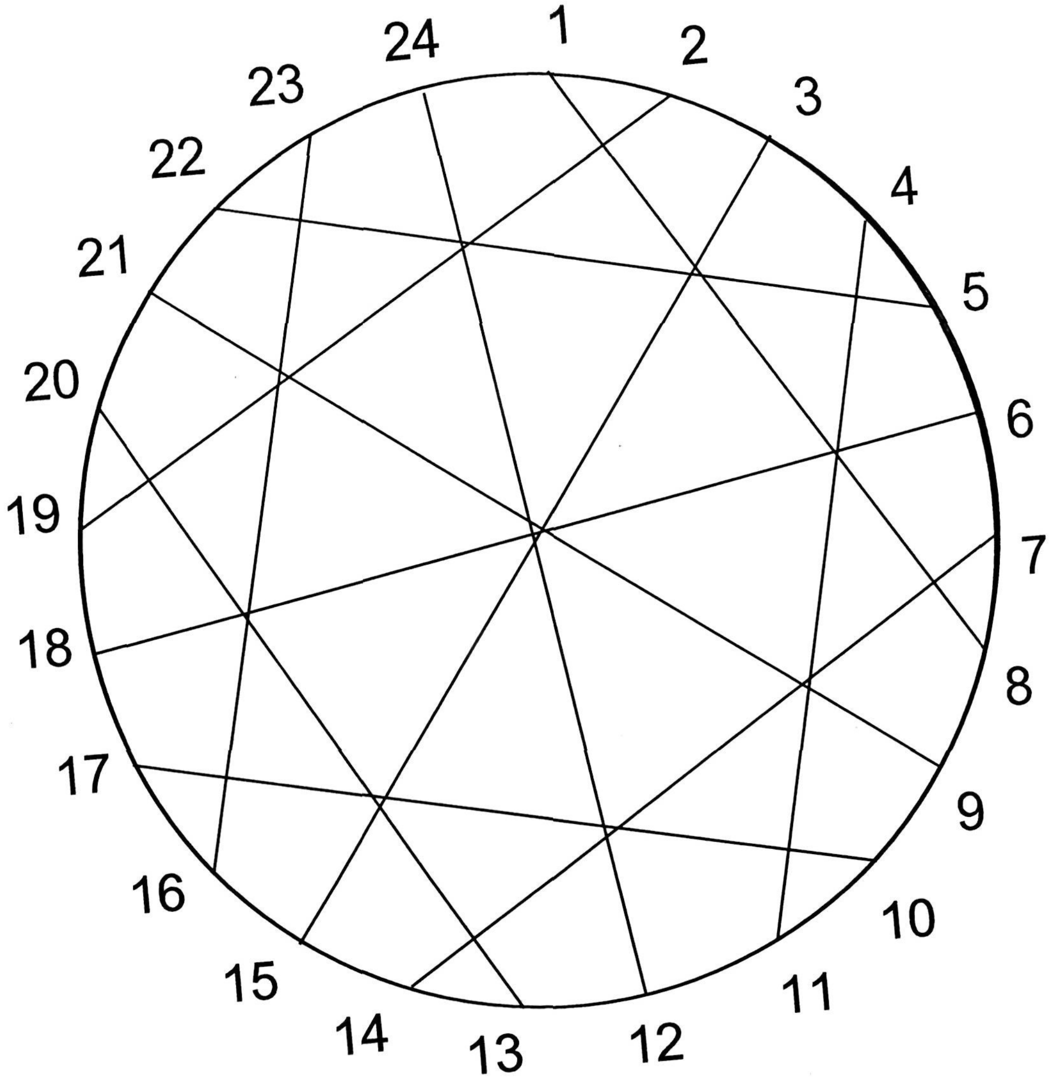
3. Harary graph $H_{4,9}$:



$D_1= \{0,3,4,7\}$, $D_2= \{1,4,5,8\}$, $D_3=\{1,3,6,8\}$ $D_4=\{2,5,6,8\}$, $D_5=\{0,2,4,6\}$

and $D_6=\{1,3,5,7\}$ are all dsd sets but none of them are independent.

4. McGee graph :

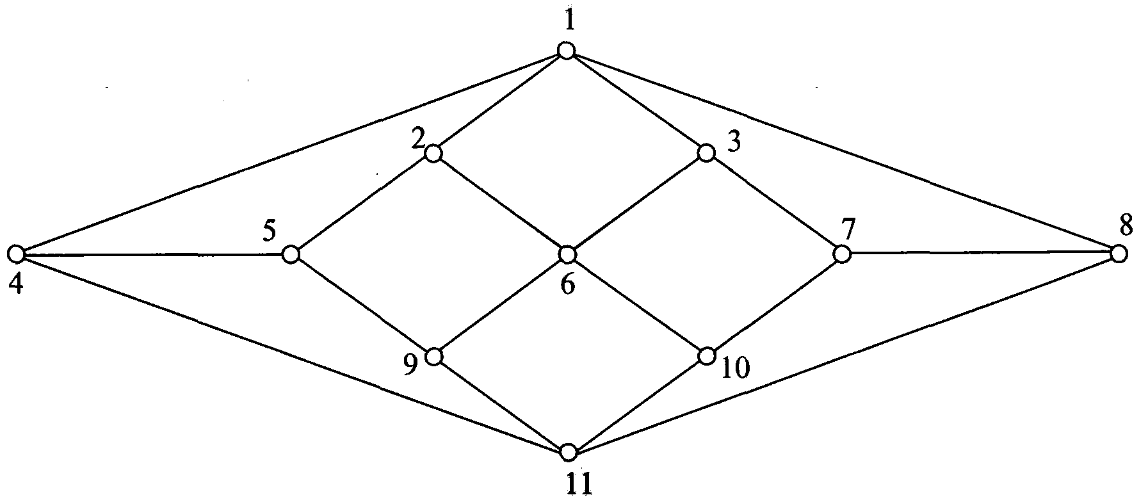


D_1= {2,4,6,8,3,7,9,10,12,19,23,13,16,22},

D_2= {1,3,5,7,11,13,15,17,19,9,21,23} and

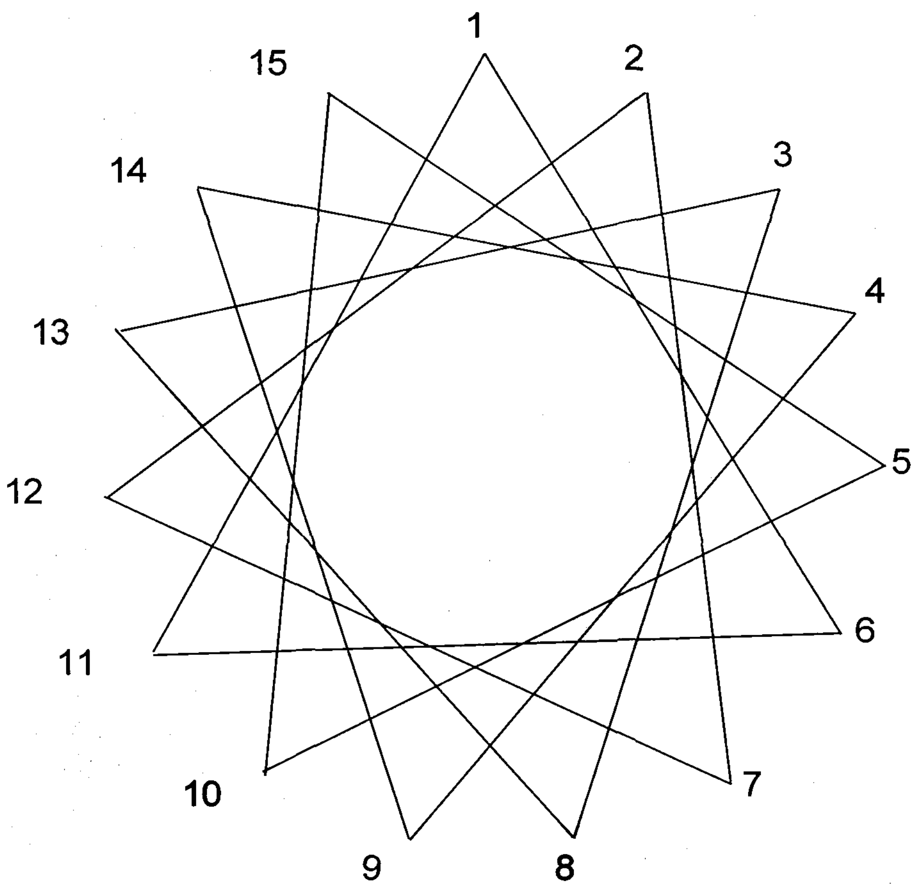
D_3= {1,4,7,10,13,16,19,15,9,22,18,24} are all dsd sets but none of them are independent.

5. Herschel graph :



D_1={1,5,7,6,11} and D_2={2,3,4,8,9,10} are independent dsd sets.

6. G (15,15) :

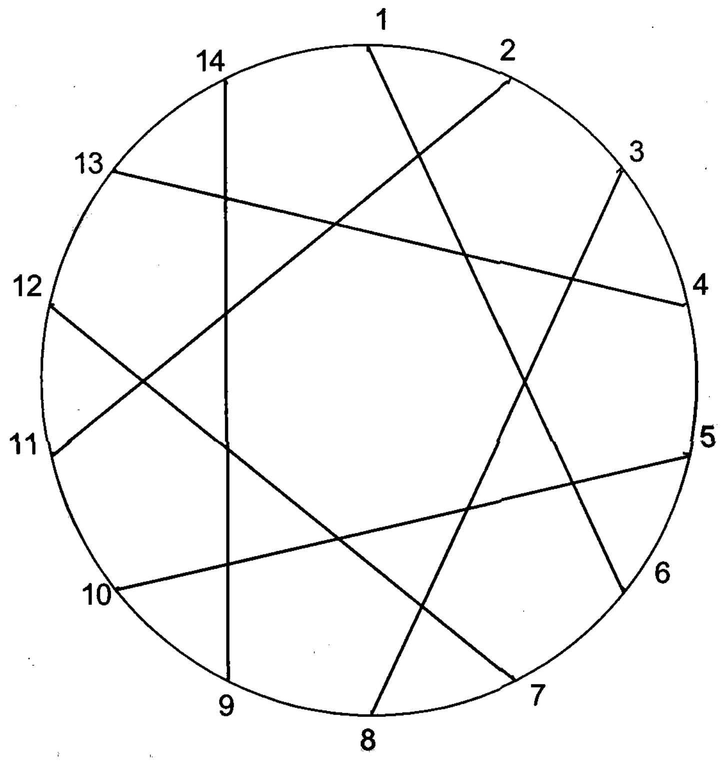


$D_1 = \{1,2,3,4,5,11,12,3,14,15\}$, $D_2 = \{1,2,3,4,5,6,7,8,9,10\}$

and $D_3 = \{6,7,8,9,10,11,12,13,14,15\}$ are dsd sets

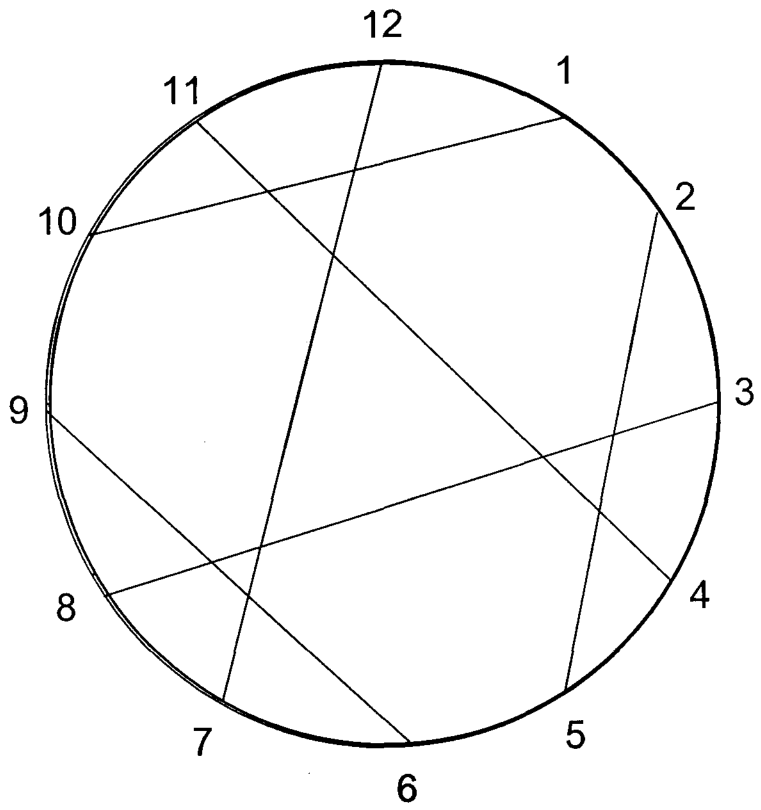
but not independent.

7. Heawood graph :



$D_1=\{1,3,5,7,9,11,13\}$ and $D_2=\{2,4,6,8,10,12,14\}$ are independent dsd sets.

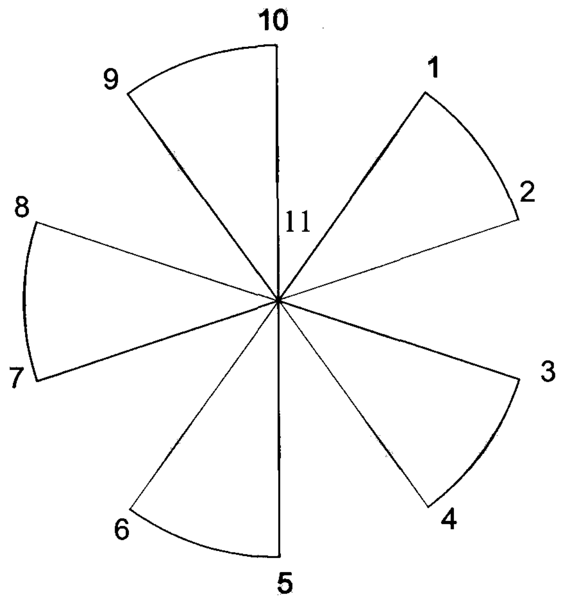
8. Franklin graph :



D_1= {1,3,5,7,9,11}, D_2={2,4,6,8,10,12} are independent dsd sets.

D_3={1,4,5,7,8,10}, D_4= {2,5,7,8,10,11} are dsd sets but not independent.

9. Friendship or windmill graph :



Here $D_1=\{2,4,6,8,10,11\}$ and $D_2=\{1,3,5,7,9,11\}$ are dsd sets but not independent.

We summarize the observed values in Tabular form:

Graph	i_{dsd}	i^s_{dsd}	β_{dsd}	γ_{dsd}	Γ_{dsd}	ir_{dsd}	ir^s_{dsd}	IR_{dsd}
Harary $H_{4,8}$	0	0	0	4	4	4	4	4
$H_{5,8}$	0	0	0	3	4	4	4	4
$H_{4,9}$	0	0	0	4	4	4	4	4
McGee	0	0	0	12	14	12	12	14
Herschel	5	5	6	5	6	5	5	6
G(15,15)	0	0	0	10	10	10	10	10
Heawood	7	7	7	7	7	7	7	7
Franklin	6	6	6	6	6	6	6	6
Wind Mill	0	0	0	6	6	6	6	6
Chvatal	0	0	0	6	6	6	6	6
Tietze	6	6	6	6	7	6	6	6
Nanogram	0	0	0	5	5	5	5	5

4.3. Connected dom-strong domination :

4.3.1. Definition :

Let G be a connected graph. Then V(G) is a connected dsd set. A dsd set D is called a connected dsd-set if <D> is connected.

The minimum cardinality of a connected dsd-set is called the connected dsd number and is denoted by γ^c_{dsd}.

4.3.2. Remark :

$\gamma_{dsd} \le \gamma^c_{dsd}$. If $\gamma_{dsd} = 2$ then $\gamma^c_{dsd} = 2$ or 3.

4.3.3. Observation :

i. γ^c_{dsd} $(P_n) = n$

ii. γ^c_{dsd} $(C_n) = n-1$

iii. γ^c_{dsd} $(W_n) = n-2$

iv. γ^c_{dsd} $(K_n) = 2$

v. γ^c_{dsd} $(K_{n,n}) = 2$

vi. γ^c_{dsd} $(T) = |V(T)| = n$.

4.3.4. Definition :

Let G be a graph without isolates. A dsd set D is called a total dsd set if for every $u \in D$ there exist $v \in D$ such that u and v are adjacent.

Every graph without isolates has a total dsd set namely V.

4.3.5. Definition :

The minimum cardinality of a total dsd set is called the total dom-strong domination number and is denoted by γ_{dsd}^{t} .

4.3.6. Observation :

i. $\gamma_{dsd}^{t}(K_n) = 2$

ii. $\gamma_{dsd}^{t}(K_{n,n}) = 2$

iii. $\gamma_{dsd}^{t}(F_n) = n$

iv. $\gamma_{dsd}^{t}(W_n) = \left\lceil \frac{n}{3} \right\rceil + 1$

4.3.7. Remark :

For any connected graph G, $\gamma_{dsd}^{t}(G) \leq \gamma_{dsd}^{c}(G)$

Proof : Since any connected dsd set is a total dsd set we have $\gamma_{dsd}^{t}(G) \leq \gamma_{dsd}^{c}(G)$ ■

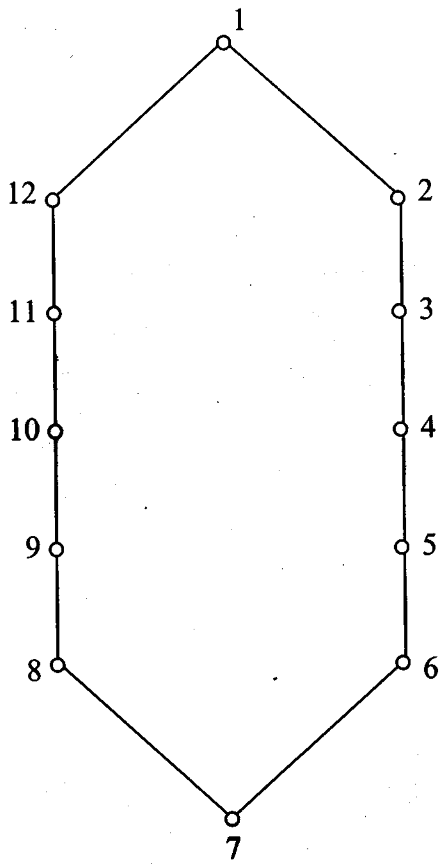
4.3.8. Result : For any graph G, $\gamma_{dsd}(G) \le \gamma_{dsd}^{t}(G) \le \gamma_{dsd}^{c}(G)$

4.3.9. Examples :

i. For K_n, $\gamma_{dsd}(K_n) = \gamma_{dsd}^{t}(K_n) = \gamma_{dsd}^{c}(K_n) = 2$

ii. For $K_{n,n}$, $\gamma_{dsd}(K_{n,n}) = \gamma_{dsd}^{t}(K_{n,n}) = \gamma_{dsd}^{c}(K_{n,n}) = 2$

iii. For C_{12} :



Here $\gamma_{dsd} = 6, \gamma_{dsd}^{t} = 8$ and $\gamma_{dsd}^{c} = 11$. So $\gamma_{dsd}(C_{12}) < \gamma_{dsd}^{t}(C_{12}) < \gamma_{dsd}^{c}(C_{12})$.

4.3.10. Remark :

$2 \le \gamma_{dsd}^{c} \le n$, and the bounds are sharp.

For K_n; $\gamma_{dsd}^{c} = 2$ and for $K_{n,n}$; $\gamma_{dsd}^{c} = n$.

4.3.11. Remark :

$$\frac{n}{\Delta(G)+1} \le \gamma(G) \le \gamma_{dsd}(G) \le \gamma_{dsd}^{t}(G) \le \gamma_{dsd}^{c}(G)$$

4.3.12. Observation :

Let H be a spanning subgraph of a connected graph G. Then there is a relation between $\gamma_{dsd}^{c}(H)$ and $\gamma_{dsd}^{c}(G)$.

4.3.13. Examples :

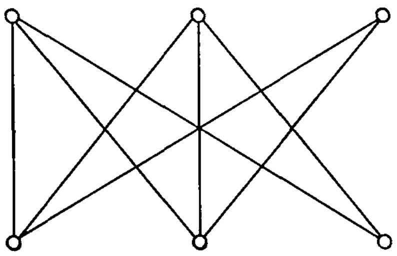
1. Let $G=K_{12,16}$ and $H'=K_{2,10} \cup K_{2,14}$ Let $V(K_{2,10})=V_1 \cup V_2$, where $|V_2|=10$ and $V(K_{2,14})= V'_1 \cup V'_2$, where $|V_2'|=14$. Join a point of V_2 to a point of V_2'. Let H be the resulting graph. Then $\gamma_{dsd}^{c}(H)=6$. H is a spanning subgraph of $G=K_{12,16}$. So $\gamma_{dsd}^{c}(G)=13$. Hence $\gamma_{dsd}^{c}(H)=6<\gamma_{dsd}^{c}(G)=13$.

   So $\gamma_{dsd}^{c}(H)<\gamma_{dsd}^{c}(G)$

2. Let G=K_n, H=C_n. Then $\gamma_{dsd}^{c}(G)=2$ and $\gamma_{dsd}^{c}(H)=n-1$

So $\gamma_{dsd}^{c}(G)<\gamma_{dsd}^{c}(H)$

3. Let G=$K_{3,3}$



Then H is a spanning subgraph of G. Hence $\gamma_{dsd}^{c}(G)=3=\gamma_{dsd}^{c}(H)$

So $\gamma_{dsd}^{c}(G)=\gamma_{dsd}^{c}(H)$.

♦ ♦ ♦

CHAPTER -V

In this chapter we proved that both the double-dominating and dom-strong dominating sets are NP-complete. It has also proved that a dsd set is NP-Complete even for Bipartite graphs. Fractional concept is also discussed, we refer to [17].

5.1. Complexity of double domination :

5.1.1. Theorem :

A double dominating set is NP-complete.

Proof :

A Double dominating set $\in$NP. For let G=(V,E) be a graph, k a positive integer and an arbitrary set $S\subseteq V$with $|S|\leq k$; It is easy to verify that in polynomial time, whether S is a double dominating set (dd-set).

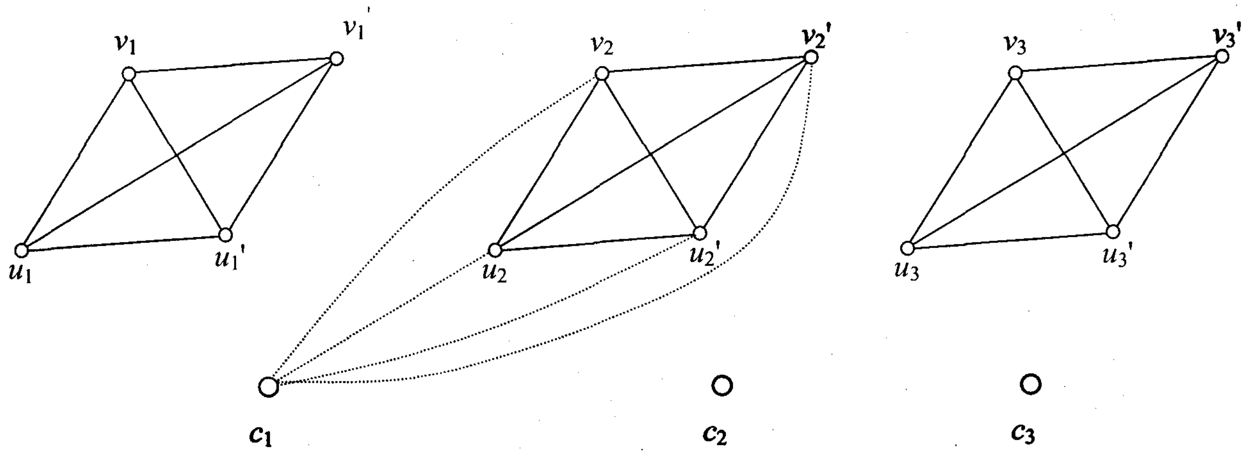
3-SAT instance :

A set $U=\{u_1,u_2,......u_n\}$ of variables and a set $C=\{c_1,c_2,........c_m\}$ of 3-element sets called clauses, where each clause c_i contains three district occurrences of either a variable u_i or its complement u_i'. For example a clause might be $c_1=\{u_1,u_2',u_4\}$.

Question :

Does C have a satisfying truth assignment that is an assignment of True or False to the variable in V such that atleast one variable in each clause in C is assigned the value True. Given an instance C of 3-SAT, we construct an instance G(C) of Double dominating set as follows :

For each variable u_i construct a K_4 with vertices labeled u_i, u_i', v_i and v_i'. For each clause $c_j = \{u_i, u_k', u_l\}$ create a single vertex labeled c_j and add edges u_ic_j, v_ic_j, $u_k'c_j$, $v_k'c_j$, u_lc_j, v_lc_j.



Suppose C is a satisfying truth assignment. Let S be a subset of V constructed as follows:

If u_i is true then u_i, v_i belong to S.

If u_i is false then u_i', v_i' belong to S.

Claim :

S is a double dominating set for the graph G(C). Suppose $u_i \in S$. Then $u_i, v_i \in S$. Therefore u_i', $v_i' \notin S$ and u_i', v_i' are dominated by u_i, v_i. If $u_i \in c_j$ then c_j is dominated by u_i, v_i. Suppose $u_i \notin S$ then $u_i' \in S$. Also $v_i' \in S$, u_i, v_i, are dominated by u_i', v_i'. If $u_i' \in c_j$ then c_j is dominated by u_i', v_i'.

Since C is a truthful assignment, every c_j contains atleast one truth variable and c_j will be double dominated by that variable in S and the corresponding v-variable in S. Therefore S is a double dominating set of the graph G(C) and |S|=2n. Conversely, suppose G(C) has a double dominating set S of cardinality $\leq$ 2n.

We must show that C has a satisfying truth assignment. Each vertex of the form v_i, v'_i must be either in S or be dominated by two vertices in S, each K_4 in G(c) must have at least two vertices in S for double domination. Therefore $|S| \geq 2n$. So $|S|=2n$.

In fact each K_4 must contain exactly two vertices of S. Therefore S contains no clause vertex c_j. But since S is a double dominating set each c_j must be dominated by two vertices in S. We create a satisfying truth assignment for C as follows : For each variable u_i assign the value true if $u_i, v_i \in S$ and false if $u_i \notin S$ and $v_i \notin S$. Let $c_j = \{x_l, x_m, x_t\}$ where x can be u or u'. Let for example, $c_j = \{u_l, u_m', u_t\}$, c_j is adjacent to u_l, v_l, u_m', v_m' u_t, v_t. Suppose c_j contain no true variable. Then u_l, u_m,' u_t are false. Therefore u_l is false, u_m is true and u_t is false. So u_l, $v_l \notin S$. u_m, $v_m \in S$ and u_t, $v_t \notin S$. Therefore c_j is not adjacent to any point of S, a contradiction. So c_j contains atleast one true variable. Hence C is a truth assignment. The graph G(C) has 4n+m vertices and 6(n+m) edges. The length of an instance of 3-SAT is 3m+n (n variable and m sets of 3 variables each). Therefore cardinality of G(C) is at most a constant times the cardinality of C. Therefore the graph G(C) can be constructed from an instance of 3-SAT in a polynomial time. Therefore double dominating set is NP-complete. ■

5.2. Complexity of Dom-strong domination :

5.2.1. Theorem :

Double –strong dominating set is NP-complete. (dom-strong dominating set)

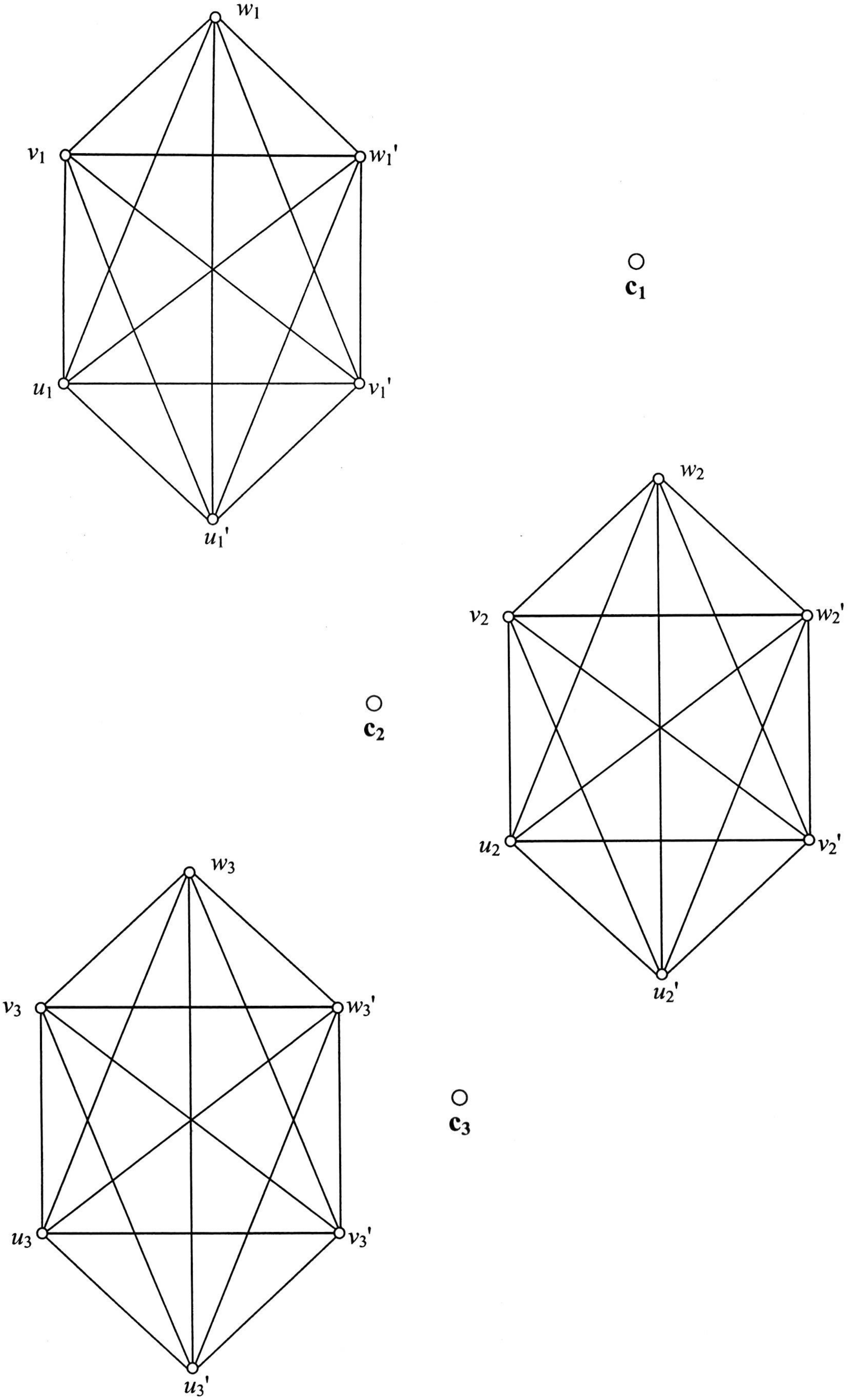
Proof :

Double –strong dominating set $\in$NP. For let G=(V,E) be a graph, k a positive integer and an arbitrary set S $\subseteq$Vwith $|S|\leq k$; It is easy to verify in polynomial time, whether S is a double-strong dominating set.

3-SAT instance :

A set U $=\{u_i,u_2.....u_n\}$ of variables and a set C = $\{c_1,c_2,.......c_m\}$ of 3-element sets called clauses, where each clause c_j, contains three district occurrences of either a variable u_i or its complement u_i'. For example a clause might be $c_1=\{u_1,u_2',u_4\}$.

Question: Does C have a satisfying truth assignment that is an assignment of true or false to the varibles in V such that atleast one variable in each clause in C is assigned value True !


w1
v1
w1'
c1
u1
v1'
u1'
w2
v2
w2'
c2
u2
v2'
u2'
w3
v3
w3'
c3
u3
v3'
u3'

Given an instance C of 3-SAT, we construct an instance G(C) of DOUBLE-STRONG-DOMINATING SET as follows :

For each variable u_i construct a K_6 with variables labeled $u_i, u_i', v_i, v_i', w_i$ and w_i'. For each clause $c_j = \{u_i, u_k', u_l\}$ create a single vertex labeled c_j and add edges $u_i c_j$, $v_i c_j$, $u_k' c_j$, $v_k' c_j$, $u_l c_j$ and $v_l c_j$. Suppose C is a satisfying truth assignment. Let S be a subset of V constructed as follows:

If u_i is true then u_i, v_i belong to S.

If u_i is false then u_i', v_i', belong to S.

Claim :

S is a double-strong dominating set for the graph G(C). Suppose $u_i \in S$. Then $u_i, v_i \in S$. Therefore u_i', $v_i' \notin S$ and u_i', v_i' are dominated by u_i, v_i. If $u_i \in c_j$ then c_j is dominated by u_i, v_i. Suppose $u_i \notin S$. Then $u_i' \in S$, $v_i' \in S$. u_i, v_i are dominated by u_i', v_i'. If $u_i' \in c_j$ then c_j is dominated by u_i', v_i'. Since C is a truthful assignment, every c_j contains at least one truthful variable and c_j will be double strong dominated by that variable in S and the corresponding v-variable in S. Therefore S is a double strong dominating set of the graph G(C) and $|S| = 2n$. Conversely, suppose G(C) has a double-strong dominating set S of cardinality $\leq 2n$. We must show that C has a satisfying truth assignment. Each vertex of the form v_i, v_i' must be either in S or be dominated by two vertices in S, each K_6 in G (C) must have atleast two vertices in S for double strong domination. Therefore $|S|\geq 2n$. So $|S|=2n$. In fact each K_6 must contain exactly two vertices of S. Therefore S contain no clause vertex c_j. But since S is a double strong dominating set each c_j must be dominated by two vertices of S. We create a satisfying truth assignment for C as follows:

For each variable u_i assign the value true if $u_i, v_i \in S$ and false if $u_i \notin S$ and $v_i \notin S$. Let $c_j=\{x_l, x_m, x_t\}$ where x can be u or u'. Let $c_j=\{u_l, u_m', u_t\}$. c_j is adjacent to u_l, v_l, u_m', v_m' and u_t, v_t. Suppose c_j contain no true variable. Then u_l, u_m', u_t are false. Therefore u_l is false, u_m is true and u_t is false. So $u_l, v_l \notin S$ $u_m, v_m \in S$ and $u_t, v_t \notin S$. Therefore c_j is not adjacent to any point of S, a contradiction. So c_j contain atleast one truth variable. Therefore C is a truth assignment. The graph G(C) has 6n+m vertices and 15 (n+m) edges. The length of an instance of 3-SAT is 3 (m+n) (n variable and m sets of 3 variables each). Therefore the cardinality of G(C) is at most a constant times the cardinality of C. Therefore the graph G(C) can be constructed from an instance of 3-SAT in a polynomial time.

So Double strong dominating set is NP-complete. ■

5.2.2. Theorem :

DSD set is NP-complete even for bipartite graphs.

Proof :

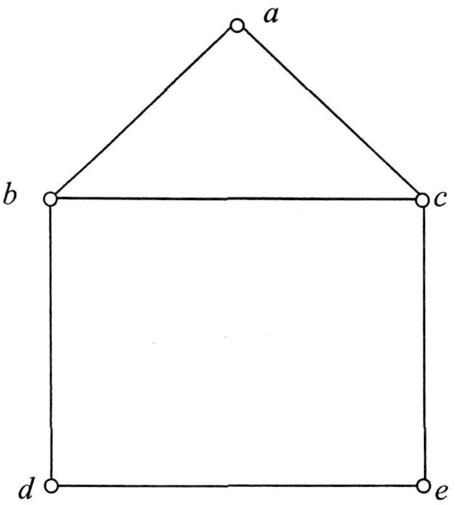
Let G=(V,E) be an arbitrary graph. Consider the graph $VV^+ = (V \cup \{x_1,x_2\}, V' \cup \{y_1,y_2, y_3\}, E^+)$ whose vertex set consists of two copies of V denoted by V and V' together with five special vertices x_1, x_2, y_1,y_2 and y_3 whose edge set E^+ consists of

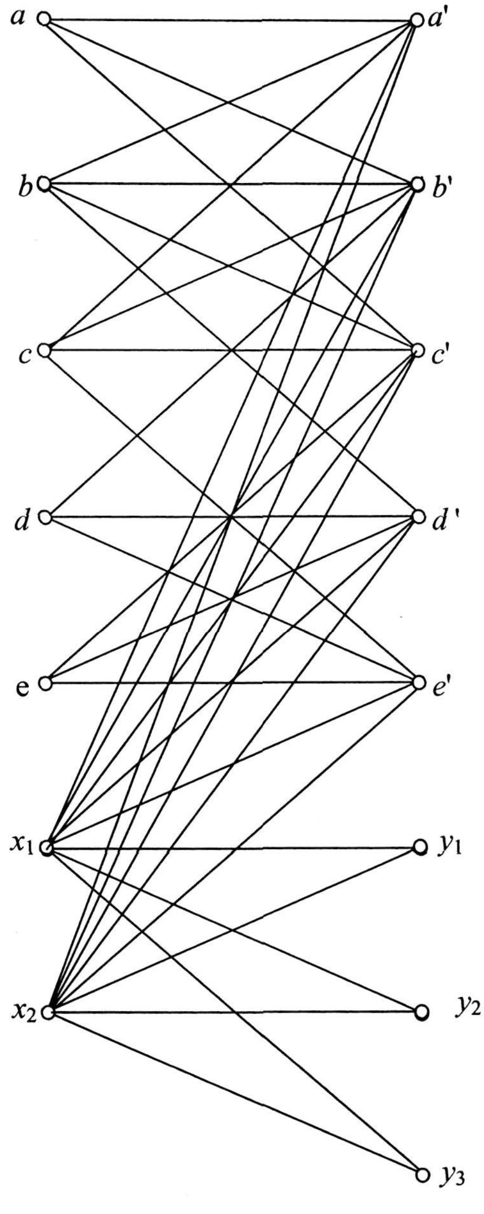
i. edges uv' and $u'v$ for each edge $uv \in E(G)$.

ii. edges of the form uu' for each vertex $u \in V$.

iii. edges of the form $u'x_1$, $u'x_2$ for every vertex $u \in V$.

iv. edges x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2, x_2y_3.





It can be observed that x_1,x_2 are of degree n+3 where n is the order of the graph. Also the degree of any point u' is ≤n-1+1+2=n+2. Therefore x_1, x_2 are strong isolates in VV^+. So any double strong dominating set of VV^+ must contain x_1, x_2. The degree of u' in VV^+ is strictly greater than the degree of u in VV^+. It can be easily observed that if $\{u_1 u_2.....u_k\}$ is a double strong dominating set of G, then $\{u_1', u_2',.....u_k', x_1, x_2\}$ is a double dominating set of VV^+. Therefore G has a dsd set of cardinality ≤ k if and only if VV^+ has a dsd set of cardinality ≤ k+2. The dsd-set of an arbitrary graph is NP-complete. Therefore DOUBLE-STRONG-DOMINATING SET is NP-complete even for bipartite graphs.

5.3. Fractional double domination:

5.3.1. Definition :

Let G=(V,E) be a simple graph. A function f:V→[0,1] is called a fractional double dominating function if for every $v \in V$, $\sum_{u \in N[v]} f(u) + f(v) \geq 2$.

5.3.2. Remark :

If D is a double dominating set of G then $\sum_{u \in N[v]} \chi_D(u) + \chi_D(v) \geq 2$.

5.3.3. Definition :

Let f be a fractional double dominating function. Define $W(f) = \sum_{v \in V} f(v)$ is called the weight of f.

5.3.4. Definition :

The fractional double domination number denoted by $\gamma_{fdd}(G)$ is the minimum weight of a fractional double domination function.

5.3.5. Remark :

For any graph G we have the following LPP:

Min $\sum_{v \in V} f(v)$, Subject to $f(N[v]) + f(v) \geq 2$, $f(v) \in [0,1]$.

5.3.6. Result :

$$\gamma_{fdd}(K_n) = 2 - \frac{1}{n} - \frac{(n-1)}{n(n+1)}$$

Proof :

Choose a vertex of K_n and fix it. Let it be u. Define f:V→[0,1] as follows :

$f(u)=\frac{1}{n}$ and $f(v)=\frac{2}{n}$, for every $v \in V(K_n)-\{u\}$

$$\sum_{w \in N[v], v \neq u} f(w) + f(v) = \frac{2(n-1)}{n} + \frac{1}{n} + \frac{2}{n} = 2 + \frac{1}{n}.$$

$$\sum_{w \in N[u]} f(w) + f(u) = \frac{2(n-1)}{n} + \frac{1}{n} + \frac{1}{n} = 2.$$

So $W(f) = 2 - \frac{1}{n}$.

Hence $\gamma_{fdd}(K_n) \le 2 - \frac{1}{n}$.

Suppose $\gamma_{fdd}(K_n) = t < 2 - \frac{1}{n} = 2 - \frac{1}{n} - s$ (say). Then there exist f:V→[0,1]

such that f(v)= $2 - \frac{1}{n} - s$ and

f(N[v])+f(v) ≥ 2 for every v∈V

or f(V) + f(v) ≥2 for every v ∈V.

or $2 - \frac{1}{n} - s +$ f(v) ≥ 2 for every v ∈V.

So $f(v) \ge \frac{1}{n} + s$ for every v∈V. That is t= f(v) ≥1+sn.

That is $2 - \frac{1}{n} - s \ge 1 + sn$

$$2 - \frac{1}{n} \ge 1 + s(n+1)$$

$$1 - \frac{1}{n} \ge s(n+1)$$

So $s \le \frac{n-1}{n(n+1)}$

Let $f(v) = \frac{1}{n} + \frac{n-1}{n(n+1)}$

Then f(N[v]) +f(v)

$$= n\left(\frac{1}{n} + \frac{n-1}{n(n+1)}\right) + \frac{n-1}{n(n+1)} + \frac{1}{n}$$

$$=1+\frac{n(n-1)}{n(n+1)}+\frac{n-1}{n(n+1)}+\frac{1}{n}$$

$$=1+\frac{(n-1)(n+1)}{n(n+1)}+\frac{1}{n}$$

$$=1+\frac{n-1}{n}+\frac{1}{n}=2$$

But $\gamma_{fdd}(K_n)\geq 2-\frac{1}{n}-\frac{n-1}{n(n+1)}$

Also $\gamma_{fdd}(K_n)\leq 2-\frac{1}{n}-\frac{n-1}{n(n+1)}$

Hence $\gamma_{fdd}(K_n)=2-\frac{1}{n}-\frac{n-1}{n(n+1)}$.

5.3.7. Theorem :

If G has n vertices and is k-regular then $\gamma_{fdd}(G)=\frac{2n}{k+2}$

Proof :

Consider the constant function f:G→[0,1] with constant value $\frac{2}{k+2}$. Then $\sum_{u\in N[v]} f(u)+f(v)=\frac{2(k+1)}{k+2}+\frac{2}{k+2}=2$. Therefore f is a fractional double domination function.

Therefore $\gamma_{fdd}(G)\leq\frac{2n}{k+2}$.

The dual of the notion of double domination function is the notion of closed neighborhood double packing function which is defined to be the function g:V $\rightarrow$[0,1] such that $\sum_{u \in N[v]} g(u) + g(v) \leq 1$.

The dual problem of the minimization of double domination function is the following problem.

Max Z*=2 $\left(\sum_{u \in V} g(u)\right)$ subject to $\sum_{u \in N[v]} g(u) + g(v) \leq 1$.

Consider the constant function f:V$\rightarrow$[0,1] with constant value $\frac{1}{k+2}$. Then $\sum f(u) + f(v) = \frac{k+1}{k+2} + \frac{1}{k+2} = 1$ and $2\left(\sum_{u \in V} f(u)\right) = \frac{2n}{k+2}$. Thus f is both a double domination and a double packing function. Therefore f is a minimum double domination function.

Therefore $\gamma_{fdd}(G)$= weight of $f = |f| = \frac{2n}{k+2}$.

Hence $\gamma_{fdd}(G) = \frac{2n}{k+2}$.

5.3.8. Corollary :

For cycle C_n, $\gamma_{fdd}(C_n) = \frac{2n}{4} = \frac{n}{2}$.

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OPEN PROBLEMS

1. Let G=(V,E) be a simple graph. Let $v \in V$. Determine the conditions for which $v \in V^+$ and $v \in V^-$.
2. Find suitable graphs which have independent dsd sets; Determine the conditions for the existence of an independent dsd set in a graph.

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Conferences / Seminars attended :

1. National Seminar on "Recent developments in concrete Mathematics" held at E.M.G.Yadava Women's College, Madurai, during 22-23 March 2002.
2. DST,AICTE-Sponsored Second National conference "On Mathematical and computational models", held at PSG College of Technology, Coimbatore, during 11-12 December 2003. **Presented a paper on "Dom-strong dominations".**
3. State level Seminar on "Graph theory and its applications", held at Lady Doak College, Madurai, during 02-03, August 2004. **Presented a paper on "k-dom-strong dominations".**
4. UGC Sponsored state level seminar on "Mathematical methods in social sciences", held at Jamal Mohammed College, Tiruchy, during 08-09, September 2004. **Presented a paper on "Dom-strong dominations of graphs and their complements".**
5. DST, CSIR, TNSCST, AICTE–Sponsored National conference on "Graphs, Combinatorics, Algorithm and applications", held at AKCE, Krishnankoil, Srivilliputhur, during 25–29, November 2004. **Presented a paper on "Excellent strong–double domination in graphs".**

6. DST, TNSCST, AICTE–Sponsored National conference on "The emerging trends in pure and applied mathematics", held at St.Xavier's College, Palayamkottai during 27-29, January 2005. **Presented a paper on "Dom-strong dominations and dsd- domatic number of a graph ".**

7. UGC Sponsored National Seminar on "Applications of Graph theory", held at Yadava College, Madurai, during 10-11, February 2005. **Presented a paper on "Split dom-strong domination in graphs".**

8. State level Seminar on "Emerging trends in mathematical methods", held at Dhanalakshmi Srinivasan College of Arts and Science for women, Perambalur, during 24-25, February 2005. **Presented a paper on "dsd–domatic number of a graph".**

9. State level Seminar on "Topology and Discrete Mathematics", held at Govindammal Aditanar College for women, Tiruchendur, on 27th February 2006. **Presented a paper on "Characterization of minimal dom-strong dominating sets".**

10. State level workshop on "Analysis of Algorithms" held at St. Xavier's College, Palayamkottai, on 4th March 2006.

11.DST sponsored "n-CARDMATH Group discussion on labelings of discrete structures and applications", held at Mary Matha Arts and Science College, Kerala, during 19-28, April 2006.

12.UGC Sponsored state level Seminar on "Recent advancements in graph theory and its applications" held at the standard fireworks Rajaratnam College for women, Sivakasi, during 18-19, August 2006.

13.UGC sponsored National Seminar on "Importance of Graph theory, fuzzy algebra and cryptography" held at Holy cross College, Tiruchy, during 24-25, January 2007.

14.State level workshop on "Recent advancements in graph theory", held at St. Xavier's College, Palayamkottai on 26th February 2007.

15.UGC Sponsored National conference on "Recent advancements in Algebraic Graph theory", held at Saraswathi Narayanan College, Madurai, during 29-31, March 2007. **Presented a paper on "A Note on complexity of double-domination and double strong domination".**

16. DST Sponsored **International conference** on "Recent Developments in Combinatorics and Graph theory", held at Kalasalingam University, Krishnankoil, Srivilliputhur, during 10-14, June 2007.

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